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ERROR ANALYSIS AND METHODS FOR ESTIMATING ERRORS IN POSITION, VELOCITY, AND ACCELERATION DATA

Data Reduction and Computing Working Group
Inter-Range Instrumentation Group
Range Commanders Council

Published May 1971 by the

Secretariat
Range Commanders Council
White Sands Missile Range,
New Mexico 88002

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FOREWORD

Mathematical models, methods and techniques which are useful and appropriate for estimating the accuracy of position (in some coordinate systems); velocity, and acceleration data are presented in this document. The development and use of the techniques discussed have evolved through the years and in some cases out of work not related to missile testing. Preparation and coordination of the material contained herein have extended over the past six years with contributions from the various organizations in IRIG. The material is not meant, or expected, to yield complete agreement as to the relative merits or importance of these procedures. However, this document is a step in the direction toward the eventual establishment of IRIG "guide lines" for recommended techniques for determining and presenting the accuracy estimates of data collected from instrumentation systems in support of operational testing. In addition, this volume will provide a source of documentation on current and available procedures as well as definitions concerning error, accuracy, and precision.

1.0 PREFACE

The Inter-Range Instrumentation Group has had as one of its goals the establishment of an IRIG standard for techniques in determining the accuracy estimates for instrumentation systems and methods of presenting these estimates. The IRIG Steering Committee assigned the task of accomplishing this goal to the Data Reduction and Computing Working Group (DR&CWG).

During the past several years DR&CWG has endeavored to investigate the many aspects of this task. Work has proceeded slowly because the formulation of standard and acceptable methods for determining accuracies is not easy. Furthermore, it is difficult to obtain agreement on the means of presenting the accuracy estimates. This work has required extensive communication and coordination on techniques in use at the various member ranges. As a result of this continued exchange of information, a common language and increased understanding have evolved. This evolution has been slow, but it now appears that sufficient agreement between member ranges and adequate technical maturity have developed.

In an effort to get some results completed in print, the DR&CWG decided to break the task into two parts. The first portion, dealing with the positional data accuracies, appeared as IRIG Document Number 104-62. The second portion, scheduled to deal with velocity and acceleration data accuracies, was written in 1963 and coordination was undertaken. However, at the 21st meeting of DR&CWG it was concluded that there was much overlap in the two documents and that it would be desirable to consolidate them since 104-62 was then at least three years old. The resulting document, 103-64, was greatly amplified and contained many new topics.

This new document basically contains the material in 103-64, with amplified material, and includes some of the newer techniques developed and used during the past few years.

2.0 INTRODUCTION

To assess the accuracy of the final result of processing data from range instrumentation is extremely difficult. To obtain agreement among the various member ranges on a standard for techniques is even more difficult. The major reasons for these difficulties stem from the variety and diversity of:

1) Various types and amounts of instrumentation used

2) Application of these instruments

3) Operational techniques employed

4) Data handling procedures

5) Mathematical and computational techniques

6) Types of requirements for reduced data

Although the member ranges have a wide variation in instrumentation and operational techniques, it is generally agreed that the measurement, X_t, collected at an instrument at time, t, is composed of a "true value" or "signal," μ_t, and an error component, ϵ_t. Thus

$$X_t = \mu_t + \epsilon_t$$

or (2.0.1)

$$\epsilon_t = X_t - \mu_t$$

The common problem in data reduction (and error estimation) is to devise mathematical models and corresponding numerical analysis techniques which will effectively separate the signal from the error in some "optimum" manner.

In discussions and conferences pertaining to error estimation, much has been said about the "true value" of a parameter. First, it is emphasized that the true value of a parameter being measured is unknown. If the true value were known there would be no need for further discussion of errors or for taking any measurements in the first place. As Cassius J. Keyser has remarked, "Absolute certainty is a privilege of uneducated minds -- and fanatics." Even though the true value cannot be known exactly, it is still a very useful concept which is used in the construction of mathematical models to represent and estimate the error in a measurement or set of measurements.

Since the true value of the signal is unknown, the approach is to devise mathematical models and experimental design techniques to give "good" or "best" estimates of the signal. In data reduction terminology, these estimates are often referred to as "computed values" or as "standards for comparison" in error analysis. The computed value, $\bar{\mu}_t$, is a numerical function of the measurements X_t or may be a simultaneous measurement from a more accurate instrumentation system or several systems. The estimate of the error is

$$\epsilon_t \approx e_t = X_t - \hat{\mu}_t. \tag{2.0.2}$$

The quantity e_t is called a residual, and applied error analysis utilizes this quantity as its basic input in estimating error characteristics.

It is important to note that the measurement X_t is a random variable and since $\hat{\mu}_t$ and e_t are numerical functions of X_t, they are also random variables. Each of these quantities are functions of time and thus make up a realization or sample from a stochastic or random process. If the observations are taken at discrete times, they form a sequence of random variables or a stochastic sequence and are commonly referred to as a discrete time series. This concept readily suggests that probability and statistics play an important role in error estimation, as well as in obtaining parameter estimates from the data reduction process.

The concept of accuracy with respect to measured and/or reduced data is closely related to the error therein, but is not identical to it. Certainly "small error" implies high accuracy or "accurate" data, and "large error" corresponds to low accuracy or inaccurate data. In general it is sufficient to state that accuracy is some function of the error distribution, and in the final analysis, accuracy itself must be estimated and based on quantities which are approximated numerically, assumed, or based on statistical estimation of parameters which characterize the error distribution.

3.0 ERROR CLASSIFICATION, ACCURACY, AND PRECISION

In accordance with Section 2.0 the error ϵ_t is estimated by the residual e_t and the accuracy of the measured and/or reduced data is estimated by some function of the residual. This function usually involves the expected value of the error. The expected errors are generally classified into two basic types, the random (noise) errors and the systematic errors. The theoretical error model is

$$\epsilon_t = S_t + N_t \tag{3.0.1}$$

where

S_t = systematic error, and

N_t = random or noise error.

The random error is nondeterministic. This means it is not described by an analytic function but must be characterized in terms of its probability distribution function. The systematic error is generally considered to be deterministic and can be represented with an analytic model. If the error distribution is thought of in terms of variation, then the systematic error comprises the "explained variation" while the random error corresponds to the "unexplained variation."

Another type of error which is not explicitly covered by the model (3.0.1), but which cannot be overlooked is "gross error." This type of error is usually due to some type of malfunction in the measurement process and is not a usable or valid measurement. For this reason data with this type of error is referred to as "bad data," "wild points," or more formally as "outlyer data." This type of error does not belong to the statistical population described by the probability distribution function of N_t and hence must be removed or edited out so it will not bias or corrupt the data reduction process. Many different types of editing techniques are used on the various ranges and are not discussed in this report. However, it is emphasized that if outlyers are ignored in the data reduction process their effect will corrupt reduced data from neighboring valid measurements and render the otherwise valid data unusable.

It is important to note that it may often be difficult to distinguish random from systematic error in a given set of measurements. This may be due to the fact that certain constraint conditions exist (geometry limitations, availability and capability of instrumentation, cost, safety, etc.) and no feasible test can be designed which will allow a reduction process to give adequate residuals; or it may be that S_t is very complicated and an adequate error model is not available; or it may be that the variance of the random error is large and the ratio of systematic error to noise error is small (error signal to noise ratio), which corrupts the estimates made of S_t. Nevertheless, in concept, and often in practice, it is possible to treat random errors statistically while the systematic errors may be estimated in a deterministic manner and corresponding corrections made to the observed data.

The combination of systematic and random error is referred to as total error, ϵ_t, and in this document the accuracy of a measurement is defined as the root mean square of the distribution of the total error. In statistical terms

$$\text{Accuracy} \sim \sqrt{E\left[(\text{total error})^2\right]} = \sqrt{E[\epsilon_t^2]} \tag{3.0.2}$$

where E (•) is the expected value of the quantity in brackets.

If the systematic error and random error are independent, then

$$\text{Accuracy} \sim \sqrt{E\left[(\text{systematic error})^2\right] + E\left[(\text{random error})^2\right]}. \tag{3.0.3}$$

Another important term which should be distinguished from accuracy is "precision." There are probably few words as loosely used by scientific personnel as precision and accuracy. Accuracy can be described as "closeness to the truth" while precision is the "closeness together" or internal consistency of a set of measurements (see Figure 1). From these descriptions we define precision as

$$\text{Precision} \sim \sqrt{E\left[(\text{random error})^2\right]}. \tag{3.0.4}$$

It is seen that accuracy requires precision but precision does not necessarily imply accuracy. If the systematic error $\gg$ random error then in this special case

$$\text{Accuracy} \sim \sqrt{E\left[(\text{systematic error})^2\right]}. \tag{3.0.5}$$

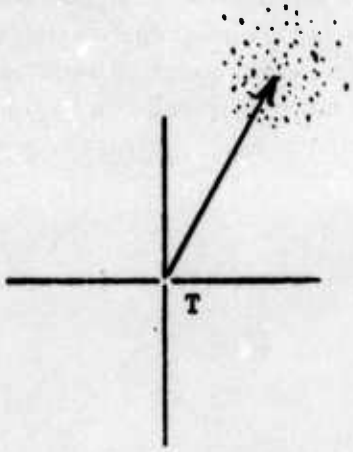
It is recognized that among the ranges there is not necessarily unanimous agreement regarding the classification of errors and the corresponding concepts of precision and accuracy. However, in the interests of practicability the errors are classified as random and systematic with the important distinction that the term "bias error" is considered to be systematic error and "constant bias error" is considered to be the constant component of systematic error.

At times it is convenient to characterize and analyze error in the frequency domain. When this is done for trajectory type data the error classification may be grouped into two categories. The high frequency category may correspond in a sense to random noise errors while the low frequency end of the spectrum may correspond to systematic errors (although not necessarily). The systematic error has a special case corresponding to zero frequency which is constant bias error. The frequency spectrum is a useful tool in the analysis of error and is discussed in more detail in Section 11.0.

To effectively estimate systematic errors with the idea of predicting what accuracy can be expected on future tests, reliance must be made on experience from past error analyses concerning the variation of systematic errors within a test and from test to test. Viewed in this manner, the systematic errors are random variables which may be characterized by a probability distribution function. An example of how the constant bias estimate from a residual distribution of an instrumentation system such as a radar will vary from test to test is illustrated in Figure 1.a.

The variation of errors within and between tests, and between instrumentation sites leads one to look for techniques which will determine whether or not significant differences exist in the error from variable factors and to estimate what effects changes in these factors have on the error. Techniques relating to this type of error analysis deal with the analysis of variance and are discussed in Section 6.3.

It is recognized that there are other descriptions which may be applied to errors. Correlation of errors in all of its detail may not be fully understood at this time. The various interdependencies of data and the added complications which arise from auto, serial, and/or time series correlation makes it unwise to attempt a full classification at this time; however, the importance of correlation between and within errors should not be ignored and a general discussion of correlation and its effects on error estimation is given in Sections 3.3 and 4.7.



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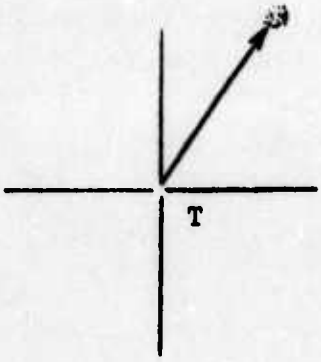
Low Precision



2 - Low Accuracy,
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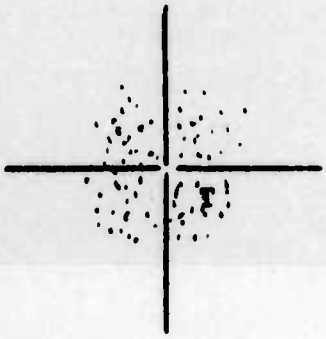
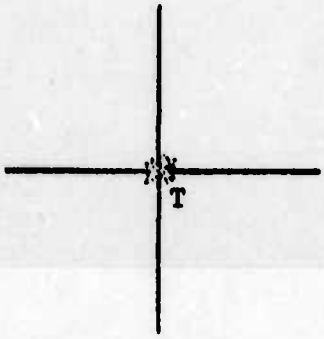
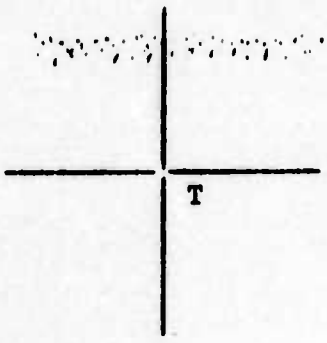


3 - Low Precision,
No Systematic Error



4 - High Accuracy,
High Precision



5 - Systematic Error

FIGURE 1. The concepts of accuracy, precision and error are illustrated. Consider T to be a target with the dots representing the results from several firings at the target

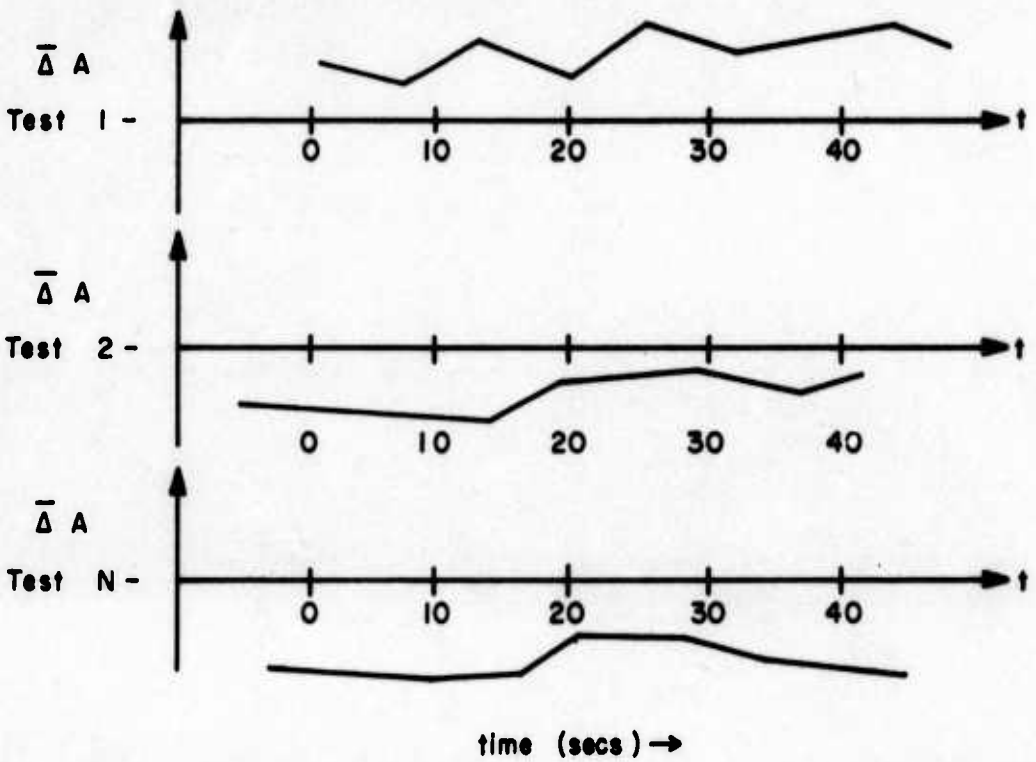


FIGURE I.a The systematic errors $\bar{\Delta}$ in azimuth A, as measured by a radar, have variations as pictured above both from test to test and as a function of time for a particular test.

3.1 ERRORS IN BASIC QUANTITIES

Thus far only errors in measurements or observations have been discussed. However, a great variety of error sources enter into the accuracy considerations for data systems used in tracking missiles. At this time and within the present state-of-the-art, it is not practical or even possible to accurately estimate the error in position, velocity, and acceleration due to each of these error sources. In an attempt to meet demands for more and more stringent accuracy requirements, great effort has been directed toward the improvement of instrumentation. However, it must be realized that even with perfect instrumentation, there are basic theoretical limitations on the accuracy which may be achieved. These limitations may introduce errors which make impossible the accuracy which is being demanded of the tracking systems.

Typical errors associated with some basic quantities are listed below. These exist quite apart from the instrumentation and place limitation or lower bound on the accuracy which may be attained. It is realized that there may be no unanimity regarding these magnitudes, or for that matter, their types, i.e., probable, absolute, etc., but they will be given as points of departure.

a)	First order, Class I surveys (distance)	10 ppm (parts/1000000)
b)	Velocity of light	1-2 ppm
c)	Index of refraction	25 ppm
d)	Changes in refraction due to rapid fluctuations in atmosphere	30-40 ppm
e)	Errors in direction cosines caused by the above fluctuations	10 ppm
f)	Overall ballistic camera accuracy	5-15 ppm
g)	Errors in reading ballistic camera film due to 1°C fluctuation in reading room temperature	5 ppm
h)	Star catalog error	1 ppm
i)	Error in 100 mile base line	1-5 feet
j)	Error in origin of national survey	50-100 feet
k)	Differences between International Spheroid and Clarke 1866 Spheroid	150-300 feet
l)	Undulations of the geoid (Δa)	50-100 feet
m)	Capability to determine relative geoid heights at widely separated stations	30 feet
n)	Voltage level (secondary standard)	100 ppm
o)	Mass	.0001-100 ppm
p)	Timing (atomic standard)	10^{-12} seconds
q)	Timing (laboratory electronics)	10^{-8} seconds
r)	Timing (instrument crystal)	10^{-6} seconds
s)	Timing correlation between separate stations	10^{-5} seconds

Some of the above values are contained in Reference (1).

It is interesting to observe the effects of the quoted survey and refraction errors when propagated into spatial position and velocity at a range of 250 miles. The resulting errors due to these sources alone are:

Position: 15 feet in horizontal coordinates
40 feet in vertical coordinates

Velocity: 0.15 foot/second in horizontal coordinates
0.40 foot/second in vertical coordinates

Obviously, these are facts which must be considered when imposing tracking accuracy requirements and in designing instrumentation systems to meet them. One is immediately faced with the operation of the law of diminishing returns.

3.2 CAUSES OF ERRORS IN INSTRUMENTATION TRACKING SYSTEMS

Once a set of residuals has been calculated with corresponding estimates of the random and systematic errors, the process of determining the cause of the errors is often extremely difficult. The diffculty encountered in determining the cause of error depends on several factors such as the ability to effectively design and control an experiment, knowledge of the variables which effect the error, correlations and effects between variables, correct assumptions concerning error model and probability distribution functions.

Some causes of random errors in measurements may be tropospheric and ionospheric scintillations, electronic noise, instrumentation wear, mechanical play, granularity, or tolerances in the measurement capabilities of the instrument, errors in computing and timing, multipath, scintillation due to echo or skin track, film reading, and so on.

Systematic errors are caused by physical limitations in achieving a true geodetic survey, electronic, optical and mechanical misalignment, servo lag, phase and frequency drift, frequency and timing bias, encoder nonlinearity, antenna droop, mislevel, lens distortion, beacon delay, miscollimation, dial eccentricity, and nonorthogonality of azimuth and elevation axes. One of the most important types of systematic error is usually considered to be constant bias or zero set error. This error is very important because it is often the most significant component of the systematic error S_t. In addition, it is the easiest component to estimate and to correct for; in fact, under actual flight conditions, the dynamics and geometry are often of such nature that the constant bias error is the only component of the systematic error which can be effectively estimated. Because of its constant nature, attempts to correct the instrumentation for this type of error can be made using static calibration tests before and after the actual flight tracking operation. Although constant bias error in the raw measurements can be considered practically constant for the duration of a single test, its propagated effects on position error may vary considerably during a single test because of sensitivity to tracking geometry. Neighboring spatial points in the time and space domains will in general have nearly constant error due to a fixed bias in misalignment. However, propagated position error would be significantly different at t+10 seconds (close range) from what it would be at t+150 seconds (slant range of several hundred miles). However, position errors due to the misalignment in the period t+140 to t+160 seconds would be expected to remain almost constant depending on geometry changes in that particular interval.

3.3 CORRELATION OF RANDOM ERROR

The random errors $N_1,\ldots,N_n$ corresponding to measurements $X_1,\ldots,X_n$ may be interrelated so that the error in the jth measurement may be dependent upon the errors in $X_{j-1}, X_{j-2},\ldots X_{j-p}$, $p \leq j$. If this relationship exists, the errors are dependent and said to be correlated. In the case of a time series this interdependence is called the autocorrelation and is estimated by means of computing the serial correlation.

From a mathematical standpoint two random variables X and Y are independent if their joint probability density

$$F(x,y) = f(x)\, g(y)$$

where f(x) and g(y) are the probability density functions corresponding to X and Y respectively. Two random variables are uncorrelated if E(XY)=E(X)E(Y). It can be shown that independence implies two random variables are uncorrelated, but not vice versa. The correlation coefficient between the random variable X and Y is defined as

$$\rho_{xy} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{[E(X - \mu_X)^2\, E(Y - \mu_Y)^2]^{\frac{1}{2}}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{3.3.0}$$

From relation (3.3.0) we see the correlation coefficient is the ratio of the covariance to the product of the standard deviations of X and Y. If a sample $X_1,\ldots,X_n$, $Y_1,\ldots,Y_n$ is taken, the correlation coefficient, ρ_{xy}, is estimated by

$$r_{xy} = \frac{(N-1)\sum_{i=1}^{N} (X_i - \bar{X})(Y_i - \bar{Y})}{N\left[\sum_{i=1}^{N} (X_i - \bar{X})^2 \sum_{i=1}^{N} (Y_i - \bar{Y})^2\right]^{\frac{1}{2}}} \tag{3.3.1}$$

If $X_1, X_2, \ldots X_n$ is a realization the autocorrelation $\rho(k)$ at lag k is estimated by the serial correlation

$$r_k = \frac{\frac{1}{N-k}\sum_{i=1}^{N-k} (X_i - \bar{X})(X_{i+k} - \bar{X})}{\left[\frac{1}{N-1}\sum_{i=1}^{N} (X_i - \bar{X})^2 \frac{1}{N-k-1}\sum_{i=1}^{N-k} (X_{i+k} - \bar{X})^2\right]^{\frac{1}{2}}}, \tag{3.3.2}$$

$$k = 0,1,2,\ldots,M<N$$

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(3.3.2) is analogous to (3.3.0) in that the autocorrelation function is the ratio of the autocovariance to the standard deviations of X_i and X_{i+k}. If we let the autocovariance estimate at lag k be R(k) and if the stochastic process is stationary, then the correlation coefficient can be replaced by the relation

$$r_k = \frac{R(k)}{R(0)} \tag{3.3.3}$$

(3.3.3) is called the normalized auto-covariance or the circular correlation coefficient. The graph of r_k is sometimes called a correlogram (see Figure 3).

The autocorrelation between errors is especially known to be present if the measurements are from an instrument whose measurements are taken with a small time increment between them. Most types of range instrumentation are of this type and as a result the autocorrelation of the errors occurs frequently.

The effects of high correlation between errors can lead one to trouble in both data reduction and error analysis. If the correlation among errors is high and does not "die out" or become small in a short time the observed data will deceptively appear to be smoother (see Figure 2), or to contain what appears to be low frequency oscillations and trends which could very easily be falsely assumed to be "signal" or valid information, and could lead one to false conclusions about the proper technique for editing and smoothing of the data. In addition, it can be shown that if high correlation exists, then estimates of the variance of the error distribution will be biased (see Section 4.7).

Several methods are used for modeling autocorrelated data. The general idea is to express the random error as a stochastic function which depends on parameters which are random variables. The most common is a pth order stochastic difference equation or pth order Markov process. The first order Markov process gives a probabilistic model of the form

$$N_i = \rho N_{i-1} + \eta_i \tag{3.3.4}$$

where η_i is an independent random variable $0 \leq \rho \leq 1$. The serial correlation coefficient corresponding to this process can be shown to be exponential, i.e., the correlation between the errors in the ith and (i+k)th measurement is ρ^k, where ρ is the correlation coefficient between adjacent measurements. Figure 2 shows independent noise (ρ=0) and correlated noise corresponding to a first order Markov process with ρ=.3, ρ=.6, and ρ=.99, respectively. Inspection of the graphs of the errors in Figure 2 shows that as the correlation becomes high, the data appears smoother and systematic trends appear in the error profile which can be very misleading.

The second order Markov process is

$$N_i = a_1 N_{i-1} + a_2 N_{i-2} + \eta_i$$

where $|a_1| < 1$, $|a_2| < 1$, and η_i is an independent random variable. In this model the autocorrelation damps out exponentially but also contains a sinusoidal oscillation. An example of FPS-16 error estimates which follow this type of autocorrelation is given in Figure 3. Note that on this figure the autocorrelation is low at a lag of 0.5 seconds between points (i.e., the correlation between points separated by 0.5 seconds is low).

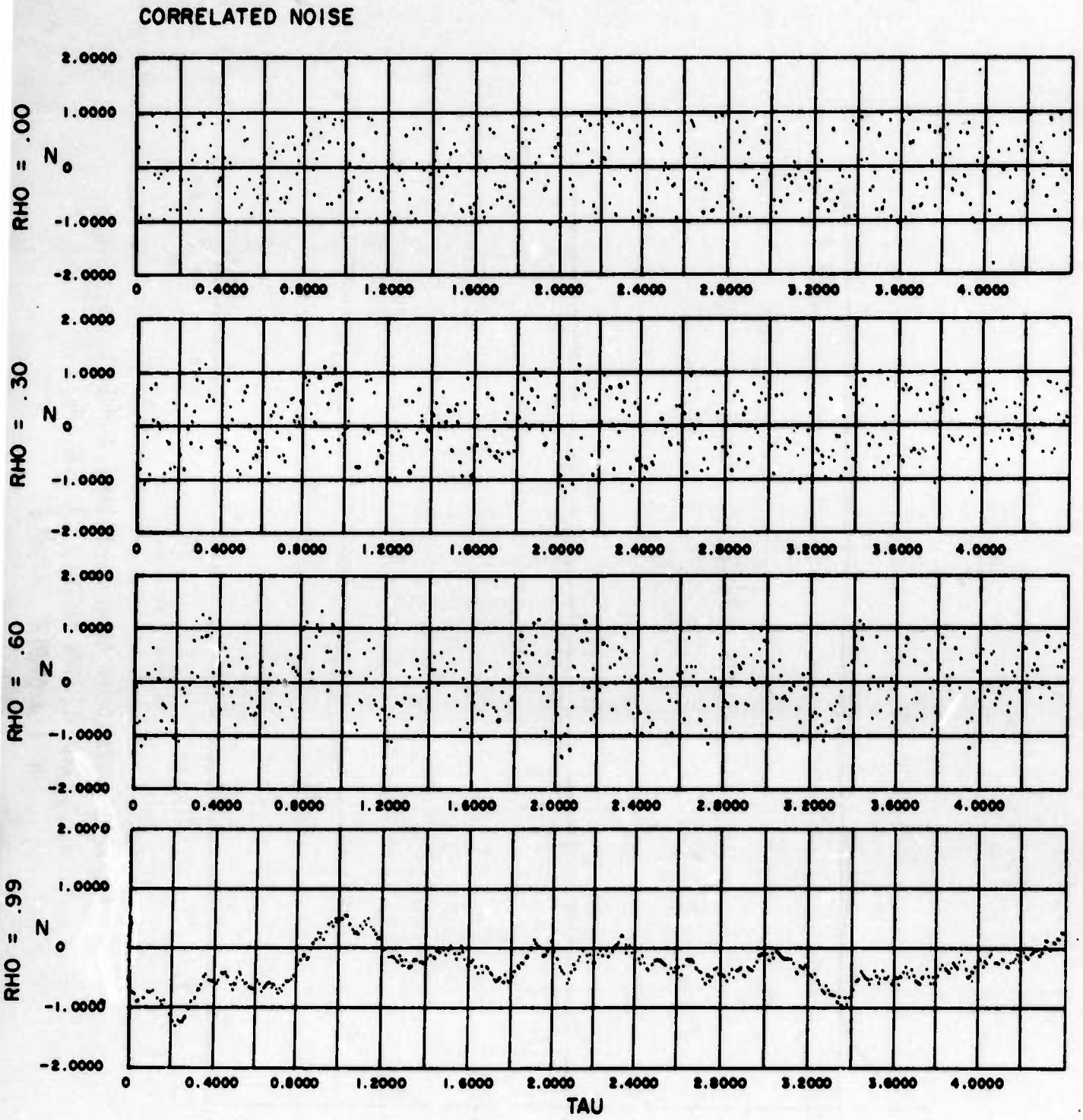


FIGURE 2 Example of uncorrelated noise (RHO = 0) and low correlated noise (RHO = .3) to very high correlated noise (RHO = .99).

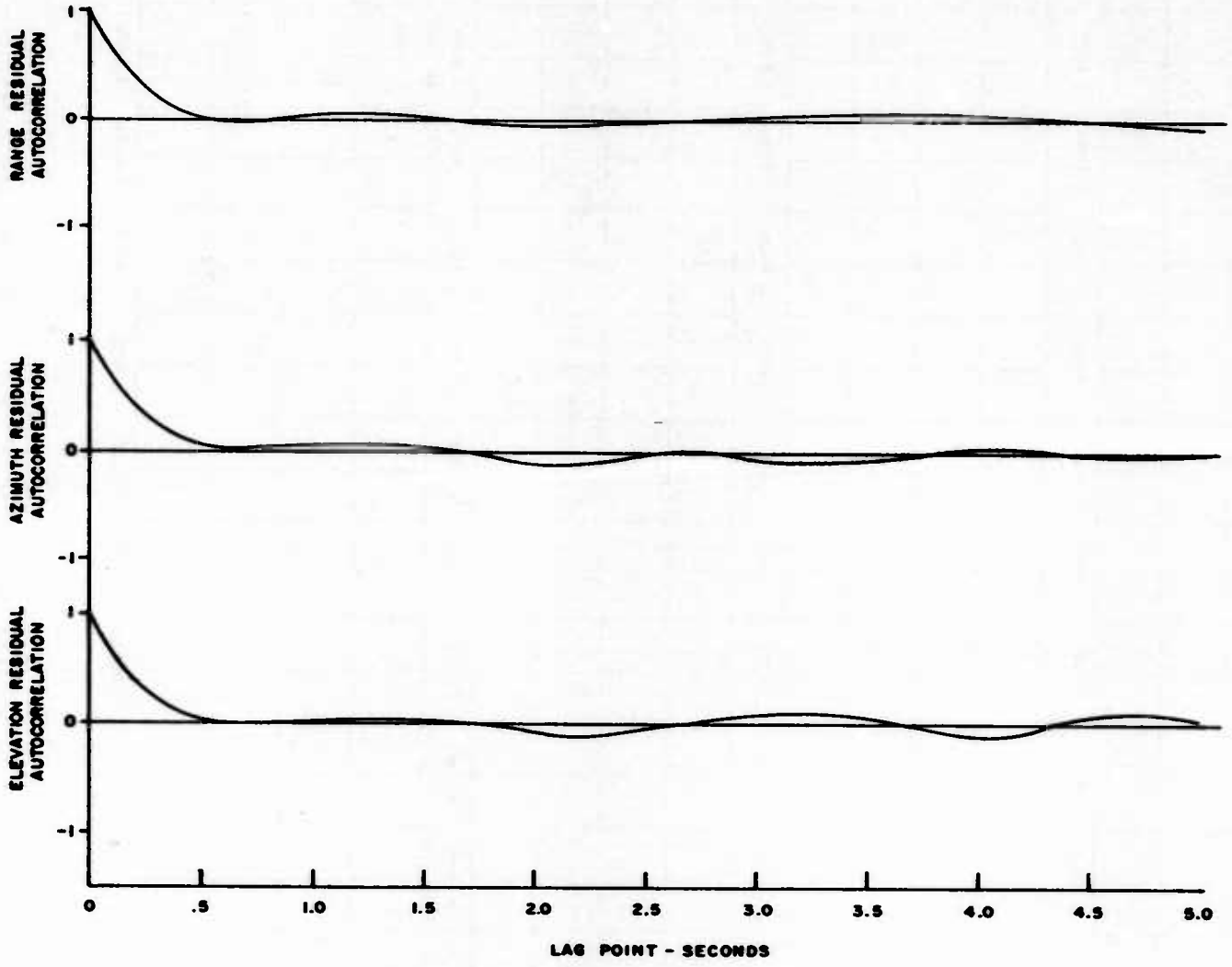


FIGURE 3 CORRELOGRAM OF AN/FPS-16 RANGE, AZIMUTH AND ELEVATION RESIDUAL DATA USING LAG OF .5 SECONDS.

4.0 METHODS USED TO ESTIMATE RANDOM ERRORS

From the previous discussion it is seen that random errors are estimated in terms of parameters which describe or characterize their behavior in terms of a probability distribution function. The parameters estimated are almost always in terms of the first and second sample moments corresponding to the residual distribution. If $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ is a set of errors, then the variance of the random error is expressed as

$$\sigma^2_{\epsilon_i} = E(\epsilon_i^2) - [E(\epsilon_i)]^2, \tag{4.0.1}$$

and if $e_1, e_2, \ldots, e_n$ is a set of n residuals from a stationary time series, which estimates this set, then the random error is estimated by

$$\hat{\sigma}_e = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (e_i - \bar{e})^2} \tag{4.0.2}$$

where

$$\bar{e} = \frac{1}{n} \sum_{i=1}^{n} e_i$$

Relation (4.0.2) is the sample standard deviation of the residual distribution. In general, the trajectory measurements themselves are changing and do not form a stationary process. However, the random error of the measurements is usually assumed to be stationary random process. This means that effective methods for removing nonstationary effects must be devised. Thus, the problem in estimating the random error is to be able to separate the nonstationary signal from the error, obtain a set of residuals, then separate the systematic error from the random error. This process is dependent on many factors including such elements as computational techniques, autocorrelation and cross correlation and assumptions concerning the analytic form of the systematic error model.

It is especially desirable to have a good estimate of the random errors for a particular system on a particular test as a function of time. This information is needed to settle such important questions as whether or not the successive errors in the measured quantity are correlated, what the optimum smoothing functions are, and what the best methods are for estimating velocity and acceleration data from position data.

Assuming that the autocorrelation and the cross correlation are zero, several methods of estimating random errors may be considered, and are given in Sections 4.1 through 4.6.

4.1 GRAPHICAL TECHNIQUES

In some instances graphical profiles of the residual distribution versus time are available for analysis. This is especially true of electronic measuring equipment where the servo feedback voltages are recorded graphically in analog form. In cases such as this, the residual profile can be inspected to see if there are systematic trends, etc. If the residual data is free of trends due to systematic error and correlation, one can select a fixed or variable sample rate and use relation (4.0.2) in estimating the random error.

If the residual is from a population characterized by the normal distribution then a much simpler method is available and is based on peak-to-peak or the sample range over a selected group or block of residuals. In selecting groups, we take k groups each containing n points. The ranges R_j which are peak-to-peak values for each jth group are estimated by

$$R_j = \max e_{ij} - \min e_{ij} \tag{4.1.1}$$

The average range $\bar{R}$ of the sample ranges is then computed, and the standard deviation is estimated from the relation

$$\hat{\sigma} = a_n \bar{R} \tag{4.1.2}$$

where a_n is the ratio $\sigma/E(R)$ for the standard normal distribution and is available in tables of the distribution of the standardized range for a normal population (see Reference 23 for tables).

Since the peak-to-peak technique is relative to the "envelope" of the residual distribution for each group or subinterval, the size of the subinterval can be selected so that trends can be removed if they exist in the data.

The graphical techniques can be very useful for "quick look" analysis but have a drawback in that they usually involve much manual analysis. When using this technique one must be careful that "wild points" are omitted since the range technique is not as stable as estimating σ by relation (4.0.2). In addition, one must be certain that the serial correlation between sampled points is not high.

4.2 LEAST SQUARES POINT ESTIMATES OF RANDOM ERRORS

This method can be most easily used when a point-by-point least squares fit to a redundant data set is being used for reduction of the data.

Examination of the discrepancies between the solutions and the observed measurements will yield information regarding the distribution of errors in the measurements. By combining many of these residuals one may estimate their standard error. A comparison of this estimate with that predicted by the a priori statistics of the adjustment gives an approximation to the error in this a priori variance data. By repeating this procedure using new estimates for the a priori variance data, an improved figure for the data error may be found.

The sample standard deviation of the space position residual at each ith time point is

$$\hat{\sigma}_i = \sqrt{\frac{\sum_{j=1}^{n} \epsilon_{ij}^2}{kn - 3}} \qquad (4.2.1)$$

where

ϵ_{ij} is the residual distance at the ith time for the jth instrument
k = number of measured parameters (R, A, E, etc.)
n = number of instruments
3 = number of parameters being estimated (X, Y, Z).

The term kn-3 represents the number of degrees of freedom available in the estimate of the variance. In the case of an optical-only solution the degrees of freedom would be 2n-3, and 3n-3 for a solution using n radars. In the case where n_1 radars and n_2 optical instruments were used, we would have $3n_1 + 2n_2 - 3$ degrees of freedom.

Position error estimates in the required coordinate system are almost always correlated and as a result, the position error distribution is characterized by the estimate of the variance-covariance or "dispersion" matrix. The variance-covariance matrix estimate for the ith time point is

$$D_i = \begin{bmatrix} \sigma^2_{x_i} & \sigma_{x_i y_i} & \sigma_{x_i z_i} \\ \sigma_{y_i x_i} & \sigma^2_{y_i} & \sigma_{y_i z_i} \\ \sigma_{z_i x_i} & \sigma_{z_i y_i} & \sigma^2_{z_i} \end{bmatrix} = N_i^{-1} \hat{\sigma}_i^2 \qquad (4.2.2)$$

where N^{-1} is the inverse coefficient matrix corresponding to the normal equation obtained by computing the partial derivatives in the least square process.

Care must be taken in applying this method, because it is capable of detecting systematic or bias error as well as random error. An examination of the time-dependent behavior of the residuals should be made to determine whether or not apparent serial correlations or trends among these errors are due to systematic causes.

If corresponding residuals from point to point display "trends," residual systematic errors are indicated which often may be estimated and removed by techniques such as a "best estimate of trajectory" (BET) solution, a multiple regression analysis or numerically filtering over a series of points. After the trends or systematic error estimates have been removed, estimates of random error may be made utilizing the new residuals which correspond to the "unexplained" or random variations about the trend.

In cases where all observations have equal weight, the standard error in the observations may be estimated by (4.2.1). For the weighted case the quadratic form of the dispersion matrix of the residuals may be used to obtain an estimate of the unit variance. That is $\sigma_e^2 = \underline{e}' W^{-1} \underline{e}$ where e is a column vector, W a weight matrix. In this case the dispersion matrix would be

$$D_i = (T_i' \; W_i^{-1} \; T_i)^{-1} \; \hat{\sigma}_i^2 \qquad (4.2.3)$$

In (4.2.3) T is the "design" matrix, T' is its transpose and the coefficient matrix of the normal equations is $N = T'T$. The matrix W_i is the weight matrix. The consistency of this estimate may be established by comparing its diagonal terms with a value of the variances known a priori.

4.3 MULTI-INSTRUMENT ESTIMATE OF RANDOM ERROR

This method, sometimes called Simon-Grubbs, estimates random error using over determination from additional independent systems. In these computations (simplified) the mathematical model for three measurements is

$$y_{i1} = x_{i1} + e_{i1},$$

$$y_{i2} = x_{i2} + e_{i2},$$

$$y_{i3} = x_{i3} + e_{i3}.$$

where y_{ij} is the measured value of the ith characteristic x_{ij} with associated measurement error e_{ij} for the jth instrumentation system. Form the three differences:

$$\Delta_{1,2} = e_{i1} - e_{i2}$$

$$\Delta_{1,3} = e_{i1} - e_{i3}$$

$$\Delta_{2,3} = e_{i2} - e_{i3}.$$

The bias error is tacitly assumed to be some fixed value; the variance estimates for the above difference equations are computed about the mean difference. If the bias is changing, it adds components of variance estimates given below:

$$s^2_{1,2} = s^2_{e1} + s^2_{e2}$$

$$s^2_{1,3} = s^2_{e1} + s^2_{e3}$$

$$s^2_{2,3} = s^2_{e1} + s^2_{e3}$$

The above equations are solved for s^2_{e1}, s^2_{e2}, and s^2_{e3}, the respective error variance estimates for instrumentation systems 1, 2, and 3. The variance of the estimated error variance for instrumentation System One is given by:

$$\text{Var}\ (s^2_{e1}) = \frac{2s^2_{e1}}{(n-1)} + \frac{1}{(n-1)}\left(s^2_{e1}s^2_{e2} + s^2_{e1}s^2_{e3} + s^2_{e2}s^2_{e3}\right).$$

For measurement Systems Two and Three, similar expressions with the subscripts permuted are appropriate.

4.4 VARIATE DIFFERENCE METHOD

The variate difference method is a technique which uses the pth order successive difference to "detrend" data and estimate the variance of the random error about the trend. It can be shown that the coefficients a_i of the linear combinations for the pth difference follow the binomial expansion law and thus the sum of the a_i^2 for the pth difference is

$$\sum_{i=1}^{p+1} a_i^2 = \frac{(2p)!}{p!(2p-p)!} = \binom{2p}{p} \tag{4.4.1}$$

Since successive differences are an analog to the time derivative it is obvious that constant linear and quadratic trends will be removed from the measured data by a first, second and third order differencing scheme. In actual practice the third difference is usually sufficient and in almost all cases the third, fourth and fifth difference estimates give equivalent results.

If $X_1, \ldots X_n$ is a set of measurements or sample from a set of independent random variables with mean μ_x and variance σ_x^2 then the linear combination

$$Y_s = \sum_{i=1}^{n} a_{si} X_i \tag{4.4.2}$$

is a random variable with mean

$$\mu_{y_s} = \left(\sum_{i=1}^{n} a_{s_i} \right) \mu_x \tag{4.4.3}$$

and variance

$$\sigma_{y_s}^2 = \left(\sum_{i=1}^{n} a_{si}^2 \right) \sigma_x^2 \tag{4.4.4}$$

When taking successive differences, the resulting difference turns out to be a linear combination of the original random variable. Each difference gives a new random variable which can be expressed as a sum of the original measurements $\{ X_i \}$. For example, the first difference is

$$Y_{1i} = \Delta^1_{x_i} = X_i - X_{i-1}, \quad a_1 = 1, \; a_2 = -1$$

For the second and third differences we get

$$Y_{2i} = \Delta^2_{x_i} = X_{i+1} - 2X_i + X_{i-1}, \quad a_1 = 1, \; a_2 = -2, \; a_3 = 1,$$

$$Y_{3i} = \Delta^3_{x_i} = X_{i+2} - 3X_{i+1} + 3X_i - X_{i-1}, \quad a_1 = 1, \; a_2 = -3, \; a_3 = 3, \; a_4 = -1$$

and so on.

From (4.4.4) and (4.4.1) the equation for the variance of the pth difference is

$$\sigma^2_{\Delta^p_x} = \binom{2p}{p} \sigma^2_x \tag{4.4.5}$$

Now the estimate of $\sigma^2_{\Delta^p_x}$ is

$$\hat{\sigma}^2_{\Delta^p_x} = \frac{1}{n-p} \sum_{i=1}^{n-p} \left(\Delta^p_{x_i}\right)^2, \tag{4.4.6}$$

so that

$$\hat{\sigma}^2_x = \frac{\sum_{i=1}^{n-p} \left(\Delta^p_{x_i}\right)^2}{\binom{2p}{p} (n-p)} \tag{4.4.7}$$

where

$\Delta^p_{x_i}$ = pth successive difference in X_i

n = number of data points in sample

p = order of the successive difference

The variate difference method is widely used since it can be applied to data when no knowledge concerning the trend in the data is available. However, caution must be used because of the assumptions which were made concerning independence of the errors. It should be emphasized that if the error is highly correlated, then the error variance estimates will be biased, and in fact usually underestimated (see Section 4.7). In addition to the correlated effects on the variate difference method the technique is also highly unstable if "wild points" or outlyers are sampled. One method of compensating for autocorrelation in the data is to compute the differences using a larger time spacing Δt between the measured X_i's. However, one must be careful that when large time increments are used, the effects of aliasing do not bias the variance estimates. A more satisfactory technique is to estimate the autocorrelation function and use this to correct the biasing effects on (4.4.7). The amount of biasing caused by autocorrelation is discussed in Section 4.7.

Since the variate difference method is based on differencing it is a useful tool in examining the frequency content of pth order differences obtained. If Δt is the time interval between successive data points, the maximum distinquishable frequency is $(1/2\Delta t)$ cycles per second and is called the Nyquist frequency. Higher order differences around the order of 4, 5 or 6 effectively examines the error contributions of frequencies between $0.82/2\Delta t$ to the Nyquist frequency. This is due to the fact that frequencies below $0.82/2\Delta t$ have been removed by differencing. If we can assume the noise is uniform over all frequencies, we may estimate the component of the variance in the interval $.82/2\Delta t$ to $1/2\Delta t$ by dividing the variance obtained using the variate difference method by four. The frequency range of $0.82/2\Delta t$ to $1/2\Delta t$ is based on the frequency response of the variate difference method. For a fourth order difference the weights are (1, -4, 6, -4, 1). If we take the Fourier transform of these weights we obtain the frequency response as given in Figure 4.

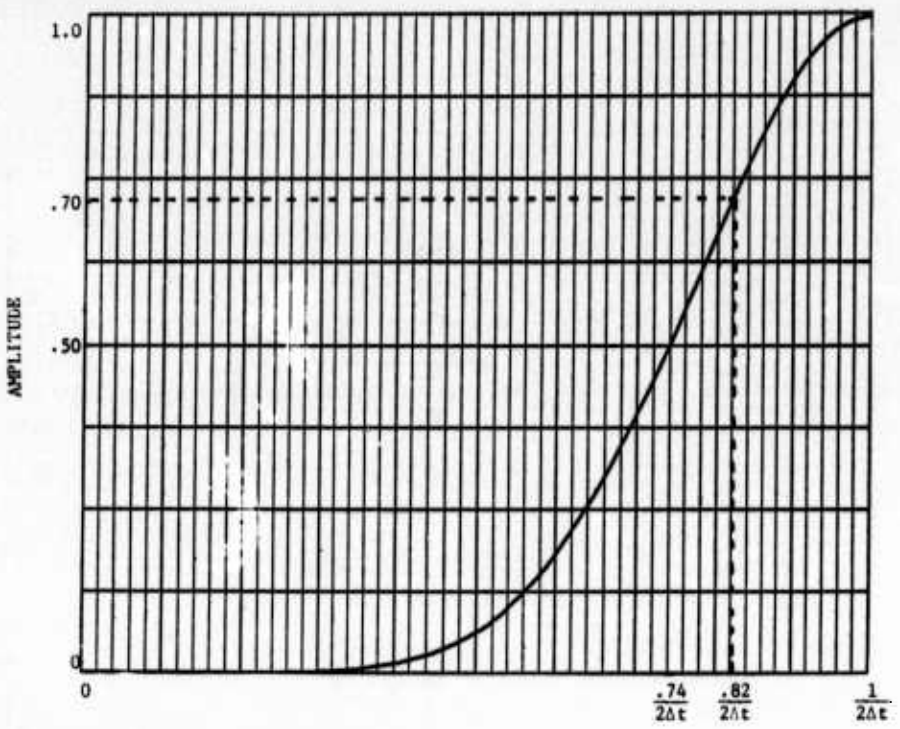


Figure 4 - Frequency response for fourth order variate difference method with lag Δt between successive points.

4.5 THE LEAST SQUARE CURVE FITTING METHOD

In the least square method a polynomial is fitted to the data points by any standard least squares curve fitting technique.

In particular, if a series of measurements, X_1, X_2,...,X_n of a parameter such as a rectangular coordinate are made as a function of time and a polynomial of degree p is fitted to this data and if the corresponding points on the curve are X_1', X_2',...,X_n', then:

$$\sigma_x = \sqrt{\frac{1}{(n - p - 1)} \sum_{i=1}^{n} (X_i - X_i')^2} \qquad (4.5.1)$$

is an estimate of the standard deviation of the random error in the measurements $X_1, X_2 \ldots, X_n$.

If p, the degree of the polynomial, is small compared to n, the number of points, the value of σ_x approximates the usual definition of RMS error, i.e.,

$$\text{RMS} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - X_i')^2} \qquad (4.5.2)$$

Estimated standard deviations are frequently plotted against the time of the mid-point of the span, and significant information concerning the variation of random errors with time or geometry can be obtained from these plots.

If the error is serially correlated then the estimates of the variance of the random error will be biased depending on the extent of the correlation. This is discussed in more detail in Section 4.7.

However, in estimating the errors by either the variate difference and/or least square curve fitting methods, it is usually assumed that autocorrelation and cross correlation are zero or very nearly so. In actuality, the following possible conditions may exist in data acquired by range instrumentation:

1. The errors in the measurements X, Y, Z at each point in time are not independent-cross correlation,

2. The errors in successive trajectory points are not mutually independent-serial or autocorrelation,

3. All measurements of a given type do not possess a common variance-heteroscedastic (versus homoscedastic),

4. Time measurements are not error free-time error.

Whenever one or all of the above conditions exist in the acquired data, more advanced and sophisticated techniques are needed to estimate the random errors in the data.

4.6 USE OF A DIGITAL HIGH PASS FILTER

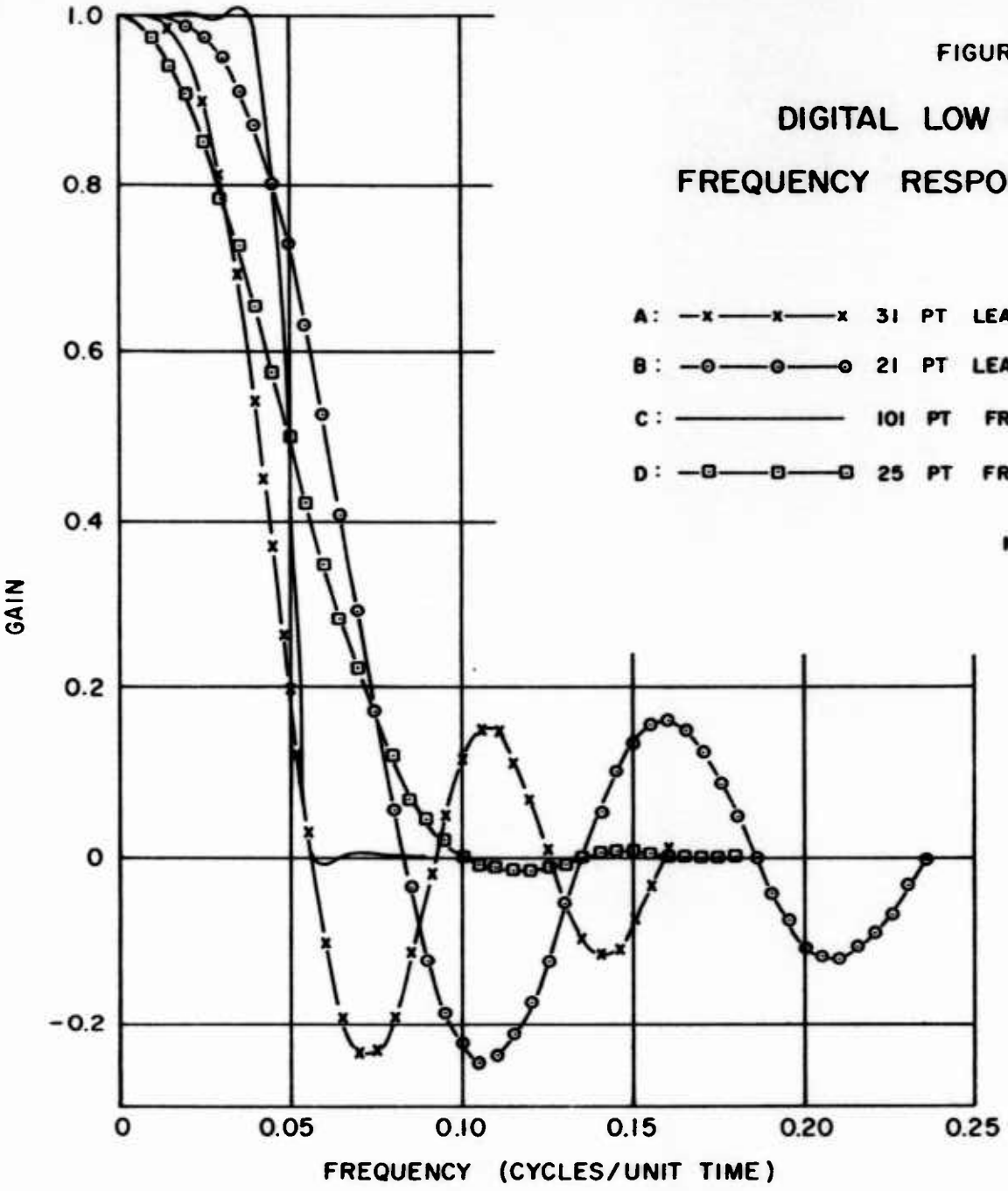
In Section 3.0 random error was characterized in the frequency domain as comprising the higher frequency portion of the power spectrum of the error distribution. If this is the case, then frequency-constraining digital filters can be constructed which will effectively separate the high-frequency components from the low at a designated cut-off frequency. This technique is often more desirable than using a least square polynomial to smooth the data and then subtracting the smooth data from the raw measurement. The reason for the desirability of the digital filter over the least-square polynomial is that the digital filter can be designed so that its frequency response has a sharper roll off with less "leakage" or side lobe effect so that undesirable frequencies do not pass through the filter. Examples of second-degree polynomial frequency responses over 31 and 21 points and digital filters with approximately the same cut-off frequencies are given in Figure 5. It is noted from the graph that the high-frequency "leakage" of the polynomial fits is as high as 25 per cent. The corresponding leakage for a low-pass digital filter over 25 points with a cutoff of 0.05 cycles per unit time is 2 per cent.

When the residual distribution is obtained from a high-pass numerical filter, the standard deviation and RMS of the distribution can be estimated using relations (4.5.1) and (4.5.2) where p is the highest degree of a polynomial which will pass through the corresponding low-pass filter without change.

FIGURE 5

DIGITAL LOW PASS FILTER

FREQUENCY RESPONSE COMPARISONS



A: —x——x——x 31 PT LEAST SQUARE QUADRATIC f_c = .04

B: —o——o——o 21 PT LEAST SQUARE QUADRATIC f_c = .06

C: ———————— 101 PT FREQ CONSTRAINED FILTER f_c = .05

D: —□——□——□ 25 PT FREQ CONSTRAINED FILTER f_c = .05

MAXIMUM LEAKAGE: A — 24 %
B — 25 %
C — 1 %
D — 2 %

4.7 EFFECTS OF AUTOCORRELATION OF THE ESTIMATION OF RANDOM ERRORS

In Section 3.3 it was pointed out that high autocorrelation makes the random error appear smoother and can lead one to false conclusions concerning reduction techniques. This is due to the fact that high correlation may appear as signal information. Because of this trait, autocorrelated random errors will bias the data, and the amount of bias usually depends on how high the correlation is.

To illustrate how autocorrelation affects data reduction and random error estimation, the different random errors illustrated in Figure 2 were superimposed on a known quadratic function, $a_0 + a_1 t + a_2 t^2$, with $a_0 = -1$, $a_1 = -1$, $a_2 = 1$, and the variance of each error distribution was $\sigma^2 = 0.01$. The results of approximating the quadratic signal with uncorrelated random error are shown in Figure 6 while the results with high autocorrelated error are given in Figure 7. Each of these figures displays a complete picture of the analysis. The first graph displays the predicted curve (line) with the raw data (signal plus noise) on the same graph as circles. The second graph displays the residuals. The information on the bottom of the figures describes how good the approximation is. A comparision of the two figures is summarized in the following table

TABLE I

Auto Correlation (rho)	Percent Error			Standard Error			Variance Estimate	Best Degree Fit
	a_0	a_1	a_2	σ_{a_0}	σ_{a_1}	σ_{a_2}		
0	1.852	3.808	1.474	.07311	.04750	.02310	.0095	2
.99	20.687	29.893	11.406	.05548	.02418	.01176	.0025	10

Table I shows that highly correlated error gives invalid results in every area. The error in the regression coefficients with rho = .99 is much higher and the estimates of the standard errors in the regression coefficients are grossly underestimated and are not consistent with the actual errors. With rho = 0 the variance estimate is very close (.0095 versus 0.01) while the variance estimate for rho = .99 is underestimated. It is noted that the best polynomial fit was correct with uncorrelated noise. However, when rho = .99 was present, the best fit was a polynomial of degree 10. The example given has shown that data containing random error with high autocorrelation can cause the following adverse effects.

1. Often makes data appear smoother.

2. May appear as signal or information.

3. Biases estimates of regression coefficients.

4. Biases estimates of variance of error distribution.

5. Biases estimates of variance of regression coefficients.

6. Biases "goodness-of-fit" tests.

The effects of correlated error on estimates of the variance when the variate difference method is used is also adverse. High correlation causes the variance estimates to be low. If the autocorrelation is a first order variate difference on each point to eliminate a quadratic trend, it can be shown that the variance estimate will be

$$\hat{\sigma}^2 = (1 - \rho)(1 - .5\rho + .1\rho^2)\ \sigma^2 \qquad (4.7.0)$$

where

ρ = correlation coefficient of lag 1

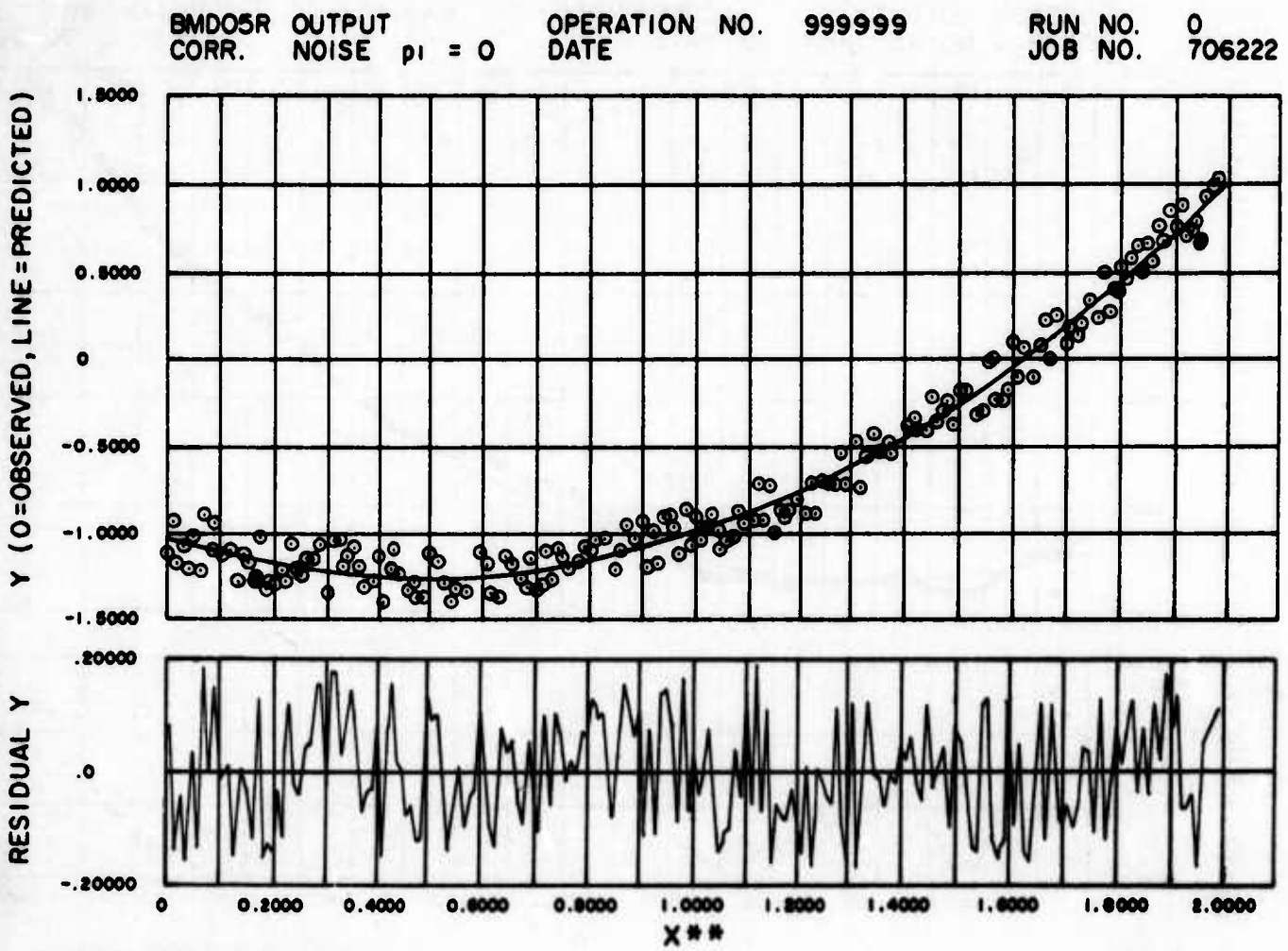
σ^2 = variance of error distribution

If we use the same technique on every other point (i.e., with a lag of 2) then the variance estimate is

$$\hat{\sigma}^2 = (1 - \sigma^2)(1 - .8\rho + .2\sigma^2)\ \sigma^2 \qquad (4.7.1)$$

The results of (4.7.0) and (4.7.1) are illustrated in Figure 8. Note from the figure that with a lag of 1 (every point) and with $\rho = 0.9$, the variance of the error is underestimated by a factor of 0.0631.

It is stressed that there are techniques which may be used to estimate and compensate for autocorrelated error. If the correlation is known a priori, it can be compensated for by "extended" or weighted least-square techniques which utilize the inverse of the autocovariance matrix to obtain a weighted estimate which compensates for correlation. If the autocovariance/correlation is not known, it can be approximated by analysis of residuals from first estimates. An iterative approximation process can be set up to obtain updated estimates of the autocovariance functions which are used for updated estimates of the regression coefficients. Information regarding the autocorrelation function may often be obtained by establishing a history of data analysis from previous operations.



POLYNOMIAL REGRESSION OF DEGREE 2

CASE NO. 1
SUBCASE NO. 0

INTERCEPT (A VALUE) 1.01652-

REGRESSION COEFFICIENTS

.96192- .98526

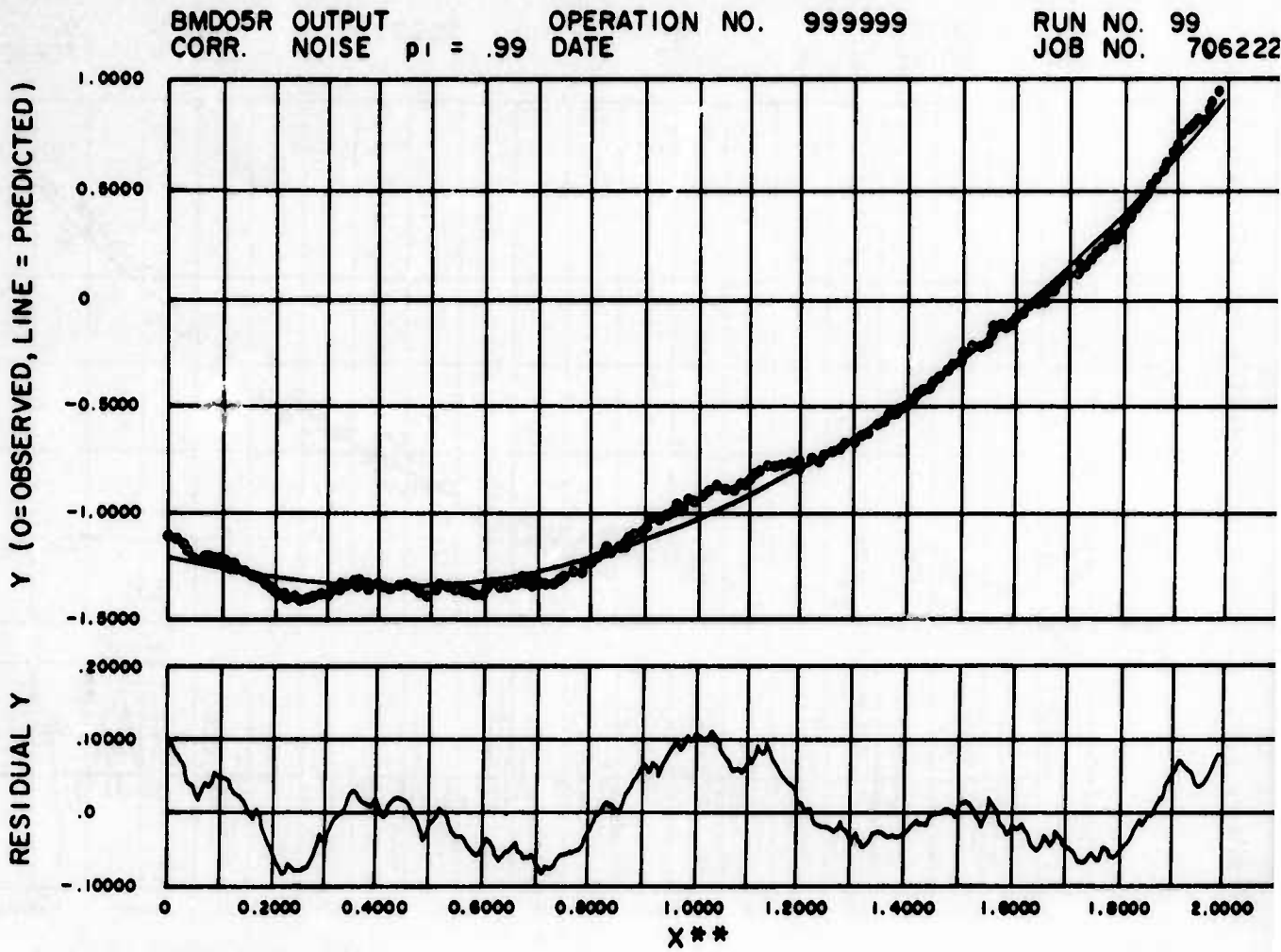
STANDARD ERROR OF REGRESSION COEFFICIENTS

.04780 .02310

ANALYSIS OF VARIANCE FOR POLYNOMIAL OF DEGREE 2

SOURCE OF VARIATION	DEGREE OF FREEDOM	SUM OF SQUARES	MEAN SQUARE	F VALUE
DUE TO REGRESSION	2	83.76314	41.87657	4414.24670
DEVIATION ABOUT REGRESSION ...	197	1.86888	.00948	
TOTAL ...	199	85.62202		

FIGURE 6 Least square regression on $y(t) = -1 - t + t^2 + N$ where N is uncorrelated noise with mean zero and variance $\sigma^2 = 0.01$.



POLYNOMIAL REGRESSION OF DEGREE 2

CASE NO. 1
SUBCASE NO. 0

INTERCEPT (A VALUE) 1.20687-

REGRESSION COEFFICIENTS

.70107- .80594

STANDARD ERROR OF REGRESSION COEFFICIENTS

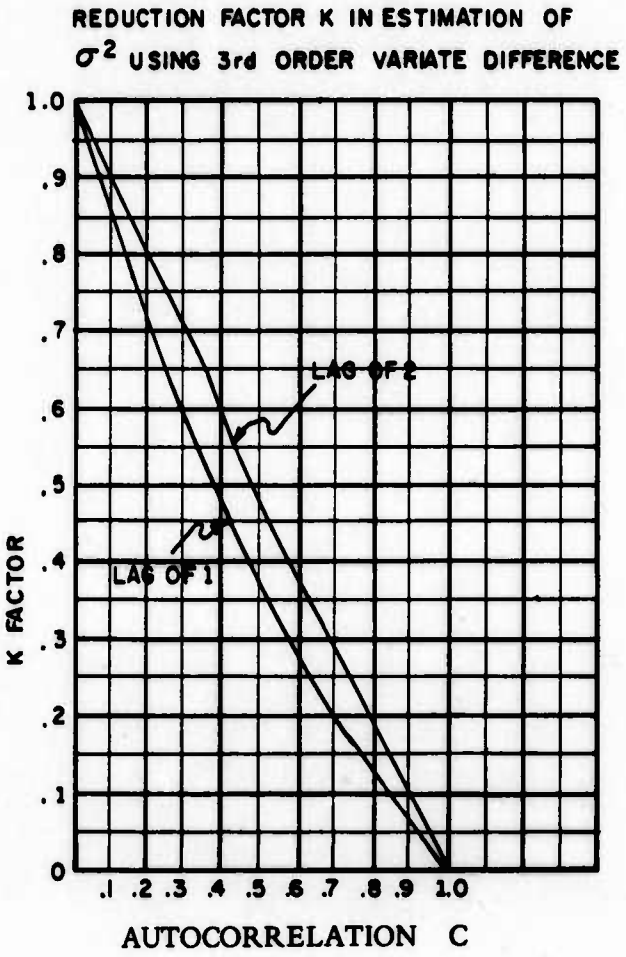
.02418 .01176

ANALYSIS OF VARIANCE FOR POLYNOMIAL OF DEGREE 2

SOURCE OF VARIATION	DEGREE OF FREEDOM	SUM OF SQUARES	MEAN SQUARE	F VALUE
DUE TO REGRESSION	2	89.13169	44.56584	18127.47094
DEVIATION ABOUT REGRESSION ...	197	.48432	.00246	
TOTAL ...	199	89.61601		

FIGURE 7 Least square regression on $y(t) = -1 - t + t^2 + N_t$ $N_t = .99\, N_{t-1} + \eta_i$ is highly correlated with $p = .99$, mean = 0 and variance $\sigma^2 = 0.01$.

FIGURE 8


REDUCTION FACTOR K IN ESTIMATION OF
σ^2 USING 3rd ORDER VARIATE DIFFERENCE
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
K FACTOR
LAG OF 2
LAG OF 1
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
AUTOCORRELATION C

5.0 INTERVAL ESTIMATION OF RANDOM ERRORS

In the previous section, techniques for point estimation of the variance, standard deviation or root mean square of the random error distribution were discussed. It should be noted that the estimates of these parameters are themselves random variables and have corresponding probability distribution functions. This fact points out that the use of sample random variables (or residuals in the case of error analysis) involves the concept of confidence, tolerance and prediction regions. In the case of one-dimensional random variables, these regions are intervals on the real line. The regions corresponding to higher-dimension random variables will be discussed in Section 8.0 after the covariance matrix has been discussed.

Interval estimation is an important concept and, in general, if we are given a sample of n residuals $e_1 \ldots, e_n$ from the population $\epsilon_1, \ldots, \epsilon_n$, the problem is to find two statistics (L, U) defining an interval which has $100\alpha\%$ probability of containing $100\gamma\%$ of the individual values in the population. Such an interval is called a tolerance interval with lower and upper tolerance limits L and U. The value of α, $0 \leq \alpha \leq 1$, is called the confidence coefficient. For example, if $\alpha = .90$ and $\gamma = .95$, we are 90% confident that at least 95% of the individual values will be in the interval (L,U).

Besides interval estimation for individual values of the population, we are interested in interval estimates for the parameters, θ, which define the probability density function of the population. We want to find statistics $L(\theta)$ such that

$$\text{Prob}\,(L(\theta) < \theta < U(\theta)) = \gamma \qquad (5.0.1)$$

where θ would be a parameter, such as the mean or variance of the distribution. In this case, the interval $L(\theta)$, $U(\theta)$ is called a confidence interval for the true (but unknown) value of the parameter. Another interval estimate is the prediction interval. In this case, we construct an interval that has the preassigned probability of containing the next observation.

5.1 THE CASE OF THE NORMAL DISTRIBUTION

It is often assumed that the random error ϵ_t is a random variable which has the normal or Gaussian distribution. This assumption may be based on experience or used because it is well known, convenient, and gives approximately correct results. It is pointed out that in the real world many random variables are not normally distributed, and this includes random errors. An example of an error which is uniformly distributed (rather than normally distributed) is round-off error. However, the importance of the normal distribution in error analysis is recognized and some of its basic uses are considered in this document.

A figure of the standard normal distribution (i.e., $\mu = 0$, $\sigma^2 = 1$) is given in Figure 9. If e_t is a sample from the population ϵ_t which corresponds to the normal distribution with mean μ and variance σ^2 (assumed known in this instance), then tolerance intervals of the form $\hat{\mu} \pm k\sigma$ corresponding to $\alpha = 1$ (absolute certainty) can be constructed. The values of k corresponding to the more widely used values of γ are given in the following table

AREAS ($100\gamma\%$) UNDER THE NORMAL CURVE IN THE INTERVAL $\mu \pm k\sigma$

k	γ
.6745	.5000
.7979	.5751
1.0000	.6827
1.6449	.9000
2.0000	.9545
3.0000	.9973

These particular values of k and γ correspond to the following error quantities

$.6745\sigma$: probable error (PE)
$.7979\sigma$: mean absolute error (ME)
σ: standard deviation
1.6449σ: map accuracy standard (MAS)
2σ: significant error
3σ: highly significant or near certainty error

Another important quantity is the standard error of the mean, or standard error of estimate. If the mean and variance of a sample are estimated from n points, then he standard error is $\sigma/\sqrt{n}$.

$$Y = \frac{1}{\sqrt{2\pi}} e^{-X^2/2}, \quad u = 0, \quad \sigma = 1$$

$X = Z/\sigma$

σ = Standard Deviation

Z = Actual Deviation

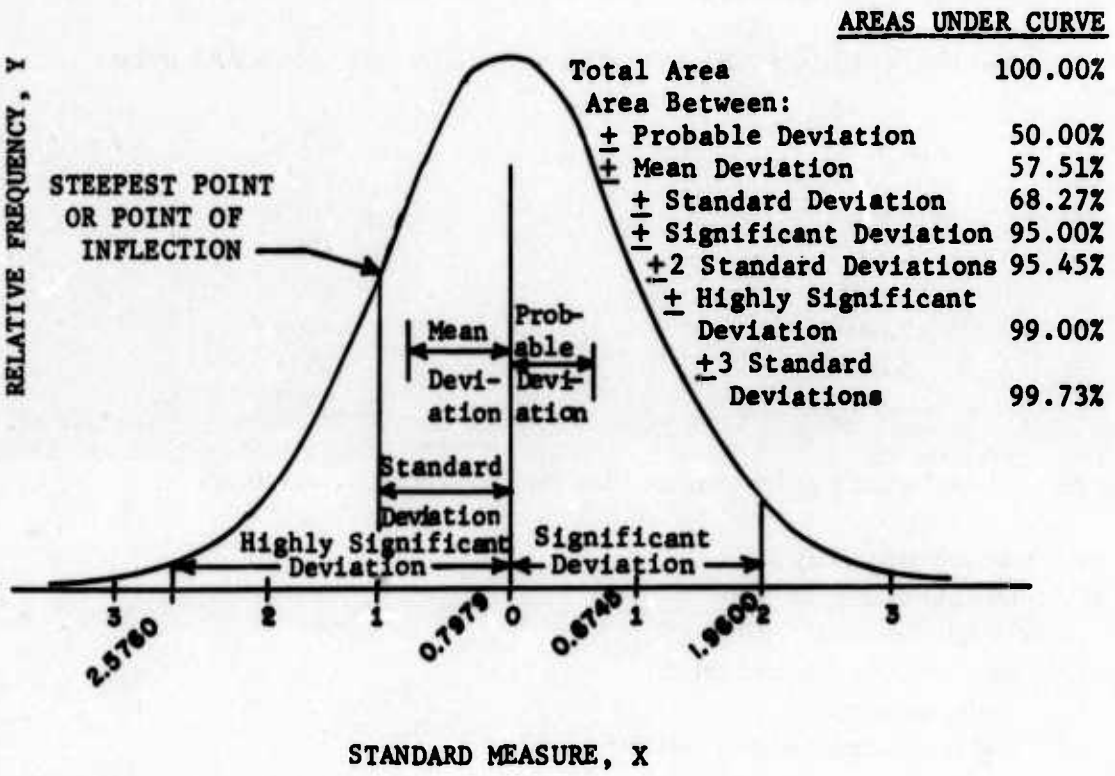


FIGURE 9. Standard Normal Frequency Distribution Curve.

5.2 CONFIDENCE INTERVALS ON ERROR PARAMETER ESTIMATES

Suppose $e_1,\ldots,e_n$ is a sample observed from the population $\epsilon_t \sim$ normal (μ,σ^2) where the population parameters μ and σ^2 are unknown. Assuming that the observations are independent, then μ and σ are estimated by the sample mean $\bar{e}$ and sample standard deviation s_e, where

$$\bar{e} = \frac{1}{n} \sum_{i=1}^{n} e_i, \tag{5.2.1}$$

$$s_e = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (e_i - \bar{e})^2}$$

$\bar{e}$ and s_e are random variables with $\bar{e} \sim$ normal $(\mu,\sigma^2/n)$ and the quantity

$$Z = \frac{(n-1)s_e^2}{\sigma^2} \tag{5.2.2}$$

has the Chi-square distribution with n - 1 degrees of freedom (i.e., Z has X^2_{n-1} distribution). It can further be shown that the statistic

$$t = \frac{\bar{e} - \mu}{s_e/\sqrt{n}} \tag{5.2.3}$$

has the "student" T-distribution with n - 1 degrees of freedom. From relation (5.2.3) we can obtain a confidence interval for μ by

$$\text{Prob}\left[t_{n-1,\left(\frac{1-\gamma}{2}\right)} < \frac{\bar{e} - \mu}{s_e/\sqrt{n}} < t_{n-1,\left(\frac{1+\gamma}{2}\right)} \right] = \gamma$$

or

$$\bar{e} \pm t_{n-1,\left(\frac{1+\gamma}{2}\right)} \frac{s_e}{\sqrt{n}} \tag{5.2.4}$$

is a $100\gamma\%$ confidence interval on μ, where t depends on the confidence coefficient α and the degrees of freedom υ which in the case of n observations correspond to (n - 1). When $t_{\upsilon,\alpha} = 1$, then the confidence interval is $\bar{e} \pm s_e/\sqrt{n}$ and the quantity $s_e/\sqrt{n}$ is called standard error of the mean or just "standard error." Values for $t_{\upsilon,\alpha}$ can be obtained from tables in most statistical texts.

In a similar manner we can get a confidence interval on σ utilizing (5.2.2). From (5.2.2) we get

$$\text{Prob}\left[\chi^2_{\left(\frac{1-\gamma}{2}\right)} < \frac{(n-1)s_e^2}{\sigma^2} < \chi^2_{\left(\frac{1+\gamma}{2}\right)}\right] = \gamma \tag{5.2.5}$$

which leads to a $100\gamma\%$ confidence interval on σ^2 which is

$$\frac{(n-1)s_e^2}{\chi^2_{\left(\frac{1+\gamma}{2}\right)}} < \sigma^2 < \frac{(n-1)s_e^2}{\chi^2_{\left(\frac{1-\gamma}{2}\right)}} \tag{5.2.6}$$

Values of χ^2_α can be obtained from Chi-square tables in statistical texts.

6.0 METHODS USED TO EVALUATE SYSTEMATIC ERRORS

The problem of evaluating systematic errors in a given instrumentation system is considerably more difficult than that of evaluating random errors, unless a suitable comparison standard is available. Unfortunately, the systematic errors can be several orders of magnitude greater than the random errors and no absolute comparison standard exists. There are, however, several approaches which can be used to evaluate systematic errors.

6.0.1 ANALYTICAL INVESTIGATION OF POSSIBLE ERROR SOURCES

This involves representing the system by as comprehensive a mathematical model as possible. Reasonable perturbation of each of the model parameters is introduced and the resultant effects of this on the data are calculated. From this analysis may be obtained an estimate of the likely range of systematic errors. This approach is limited by the store of scientific knowledge, which makes it extremely difficult to construct a sufficiently comprehensive mathematical model of the system. This would appear to be especially true of propagation anomalies which particularly affect electronic systems.

6.0.2 COMPARISON OF OBSERVATIONS FROM DIFFERENT SYSTEMS

If it is known that System I is significantly more accurate in an absolute sense than System II, then an estimate of the systematic error in II can be made by noting the discrepancies between observations of the two systems. The estimate of the total error in System II is made by computing the difference

$$\Delta = (\text{System II} - \text{System I})$$

at each time point. The difficulty with this approach lies in selecting a suitable standard. Emphasis must be placed upon the fact that upper bounds on possible systematic errors in the standard must be known, and these bounds must be narrow compared with reasonably lower bounds for the systematic error in the system being investigated. Thus, for the calibration standard it is absolute accuracy, not relative or internal accuracy, which is of prime importance. This fact makes it unprofitable to compare systems indiscriminately. Also, it is necessary to employ a mathematical model of the system being treated in order to transform the standard data and propagate any error in the standard into the tested system.

The observations collected and used for error analysis may be from actual operational tests or from tests which are specifically designed for the purpose of error analysis. Tests of the latter type are called calibration tests. The most simple, economical, and frequently used calibration technique is the "static" calibration test. This utilizes calibration equipment and objects as "line-ups," boresight towers, and stars. In most instances, however, the static calibration test is not effective in estimating the many errors encountered during an actual operation. The reason for this is that most of the systematic error components are dynamic in nature and depend upon variables such as distance, tracking rates, atmospheric conditions (temperature pressure, humidity, etc.), and the direction in which the instrumentation system is pointing. As a result, dynamic calibration tests utilizing a good test design and a standard with sufficiently small error provides a better estimate of the systematic error.

6.0.3 ANALYSIS OF MEASURING RESIDUALS FROM OVER–DETERMINED LEAST SQUARES SOLUTIONS

When a given instrumentation system provides a redundancy of data, a least square solution may be performed. The adjustment leads to a set of measuring residuals (or corrections) for each trajectory point. The criterion for adjustment is the principle of maximum likelihood. If these residuals indicate, from point-to-point, a randomness about zero, there is good indication that the systematic error is small. If the residuals do not display this randomness about zero, one must conclude that the original observations are biased or have systematic error. A total error computed from the residuals gives an estimate of the upper limit of the systematic error in the observations.

6.0.4 ANALYSIS OF RESIDUALS FROM A BEST ESTIMATE OF TRAJECTORY

When there is simultaneous track with more than one instrumentation system, a Best Estimate of Trajectory (BET) can be computed. This trajectory and the residuals of the individual systems referenced to it can then be analyzed to determine estimates of the systematic errors. The BET is a powerful method of estimating systematic errors in both the reduced data and the measured parameters. There are several methods of obtaining a BET, depending upon the criteria for test. The most powerful of the methods developed to date involves the use of mathematical models of the errors of the individual systems. In all the applications of BET to error studies, the basic idea is that, according to the criterion agreed upon, a best estimation of what the observations should have been is made and then compared with the observed data.

6.0.5 MULTILATERATION

As in the case of BET, when redundant data from several instruments is available a multilateration technique which uses inputs from several different sources such as doppler, pulse radar data, and inertial guidance data can be used. This technique has the error model design as part of the solution and attempts to "weight out" the effects of error sources by associating each source with an uncertainty which is estimated from a priori testing and is updated by the multilateration solution. If error sources are correlated, they adversely affect the solution and must be taken out, using estimates from calibration performed prior to the test. Once this is done, estimates of the magnitudes of all the error model coefficients are made simultaneously employing a maximum likelihood technique, which is mechanized using the Kalman filter approach.

6.1 ERROR MODELS

In Section 3.0 it was stated that systematic errors are deterministic. This means the systematic error S(t) can be represented analytically in terms of a mathematical model. Since the random error is nondeterministic, it must be represented by some probability model. In general, one may set up error models which fall into the following categories:

I. Deterministic

Parametric estimation—an analytic model which describes the error in terms of parameters which are the components of the error.

Nonparametric estimation—a numerical model which approximates the error numerically, i.e., as a sequence of numbers such as the output from a digital filter.

II. Nondeterministic

Parametric estimation—a stochastic equation depending upon parameters which are random variables, e.g., the expression of a random process by a pth order Markov Process. If the process is stationary it can be characterized in terms of parameters which index its probability density function.

The most simple systematic error model would be

$$S(t) = \mu \tag{6.1.1}$$

In this case the systematic error is a constant bias, "offset" or "zero set" error. If S(t) is estimated by a sequence of n Δj's, then

$$\mu \triangleq \bar{\Delta} = \frac{1}{n} \sum_{j=1}^{n} \Delta j \tag{6.1.2}$$

The Δj's also contain random error and the averaging in (6.1.2) separates μ from N(t). The estimate of the random error parameter would be

$$\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{j=1}^{n} (\Delta j - \bar{\Delta})^2}. \tag{6.1.3}$$

If the error components are independent, then the error model can be expressed as a linear combination of independent variables, i.e.,

$$\varepsilon_i = \sum_{j=1}^{k} \alpha_j x_i + \eta_i \tag{6.1.4}$$

The error model given in (6.1.1) can be expanded to include terms which describe various error components. A good example is a typical error model frequently used for the AN/FPS-16 radars. The error models for range, azimuth, and elevation are given below.

$$\Delta R_j = a_0 + a_1\dot{R}_j + a_2\ddot{R}_j + a_3\dddot{R}_j + a_4 f(S/N) + a_5 R_j + a_6 R_j\dot{R}_j \qquad (6.1.5)$$

$$+ a_7 \csc E_j + a_8 \sin \frac{2\pi}{6000}\dot{R}_j + a_9 \cos \frac{2\pi}{6000}\dot{R}_j + a_{10}\frac{X_j}{R_j}$$

$$+ a_{11}\frac{y_j}{R_j} a_{12}\frac{z_j}{R_j} + N_j$$

where

a_0 = constant bias
a_1 = timing delay
a_2 = acceleration servo lag
a_3 = jerk servo lag
a_4 = beacon delay
a_5 = oscillator drift or scale factor
a_6 = time dilation
a_7 = residual refraction
a_8 = resolver nonlinearity
a_9 = resolver nonlinearity
a_{10} = x survey error
a_{11} = y survey error
a_{12} = z survey error
N_j = random error

$$\Delta A_j = b_0 + b_1\dot{A}_j + b_2\ddot{A}_j + b_3\dddot{A}_j + b_4 \sin A_j \tan E_j + b_5 \cos A_j \tan E_j \qquad (6.1.6)$$

$$+ b_6 R_j\dot{A}_j + b_7 \sec E_j + b_8 \tan E_j + b_9 \sin A_j + b_{10} \cos A_j$$

$$+ b_{11}\frac{1}{R_j} \cos A_j \sec E_j + B_{12}\frac{1}{R_j} \sin A_j \sec E_j + N_j$$

where

b_0 = constant bias
b_1 = timing bias
b_2 = acceleration servo lag
b_3 = jerk servo lag
b_4 = mislevel
b_5 = mislevel
b_6 = time dilation
b_7 = collimation
b_8 = nonorthogonality
b_9 = encoder nonlinearity
b_{10} = encoder nonlinearity
b_{11} = x survey
b_{12} = y survey
N_j = random error

$$\Delta E_j = c_0 + c_1 \dot{E}_j + c_2 \ddot{E}_j + c_3 \dddot{E}_j + c_4 \sin A_j + c_5 \cos A_j + c\, R_j \dot{E}_j \qquad (6.1.7)$$

$$+ c_7 \cot E_j + c_8 \sin E_j + c_9 \cos E_j + c_{11} \frac{1}{R_j} \sin A_j \sin E_j$$

$$+ c_{12} \frac{1}{R_t} \cos A_j \sin E_j + c_{13} \frac{1}{R(t)} \cos E_j + N_j$$

where

c_0 = constant bias
c_1 = timing delay
c_2 = acceleration servo lag
c_3 = jerk servo lag
c_4 = mislevel
c_5 = mislevel
c_6 = time dilation
c_7 = residual refraction
c_8 = encoder nonlinearity
c_9 = encoder nonlinearity
c_{10} = x survey
c_{11} = y survey
c_{12} = z survey
N_j = random error

In the linear error models the dependent variable is the total error residual, and the independent variables are usually assumed to be uncorrelated and are determined by the physics or geometry of the situation. The independent variables are assumed to be known without (or at least with negligible) error. Since the independent variables depend on the physics and geometry of a particular test for a certain instrumentation system, it is important to stress that the error model depends not only on analysis of the physics involved, but on the design of the test itself. For example, if one had a near-perfect standard for comparison for obtaining good error estimates, one could not obtain a valid estimate of the mislevel error coefficients if the test design failed to enable the instrumentation system to traverse less than 180^o in azimuth. Also, one could not estimate scale factor error if the range remained approximately constant. It is obvious that in order to estimate dymanic errors one must have a corresponding dynamic test design.

One proposed test for dynamic error estimation involves the use of a calibration satellite. This type of vehicle would allow tracking in virtually every quadrant over a wide variety of rates, look angles, and positions. Further, its trajectory can be predicted very accurately using the equations of motion as a model. The orbital parameters can be determined by BET techniques, using world-wide networks of instrumentation systems. Many problems would still exist in a program of this type, such as unmodeled parameters in the orbital model, nonuniformity in the earth's gravitational field, etc. However, during certain satellite passes and combinations of passes, the effects of such errors can be minimized. For further detailed information on the use of a calibration satellite, see Reference 24.

6.2 REGRESSION ANALYSIS

The linear error models discussed in the previous section are generally approximated utilizing multiple linear regression techniques. The techniques for regression analysis are discussed in this section and the techniques for evaluating the validity of the analysis are discussed in Section 6.3, Analysis of Variance.

When an acceptable standard is available and $e_1, e_2, \ldots, e_n$ obtained they are assumed to be a realization from the process

$$\epsilon_i = \alpha_o + \alpha_1 X_{i1} + \alpha_2 X_{i2} + \ldots + \alpha_k X_{ik} + \eta_i \tag{6.2.1}$$

$i = 1, 2, \ldots, n$. The α's are unknown parameters and η_i is the random error. η is assumed to be a stochastic process whose elements are homoscedastic, independent, with mean $\mu_\eta = o$ and variance σ_η^2. In matrix form (6.2.1) is

$$\underline{\epsilon} = X\underline{\alpha} + \underline{\eta} \tag{6.2.2}$$

where $\underline{\epsilon}$ and $\underline{\eta}$ are n x 1 column vectors, $\underline{\alpha}$ is a (k+1) x 1 column vector and X is an n x (k+1) matrix of known elements

$$X = \begin{bmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1k} \\ 1 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{nk} \end{bmatrix} \tag{6.2.3}$$

The matrix X consists of the n values of the k independent variables and is called the design matrix. In the analysis and design of experiments, the independent variables are called factors, the dependent variable (in this case the error) is called the response variable or yield. The n values of the independent variables are called levels.

In the standard least squares analysis of (6.2.2), estimates $\underline{a}$ are made of the parameters α such that the sum

$$S = \sum_{i=1}^{n} [e_i - a_o - a_1 X_{i1} - \ldots - a_k X_{ik}]^2 \tag{6.2.4}$$

is minimized. This done by taking

$$\frac{\partial S}{\partial a_o} = \frac{\partial S}{\partial a_1} = \ldots = \frac{\partial S}{a_k} = 0. \tag{6.2.5}$$

The system (6.2.5) gives the normal equations for $\underline{a}$. These are

$$(X^T X)\underline{a} = X^T \underline{e} \tag{6.2.6}$$

and it can be shown that $(X^T X)$ is a positive definite matrix and has an inverse. Thus

$$\underline{a} = (X^T X)^{-1} X^T \underline{\epsilon} \tag{6.2.7}$$

The a's are called regression coefficients and are least square estimators of the parameters $\underline{\alpha}$. Since the a's are dependent upon the e_i's they themselves are random variables and are characterized by a kth dimensional probability density function. Since the system of normal equations was linear, the variance of the regression coefficients is estimated by the sample covariance matrix:

$$\text{Var}(\underline{a}) \doteq \theta^2_{\underline{a}} = [X^T X]^{-1} S_N^2 \tag{6.2.8}$$

where

$$S_N^2 = \sum_{i=1}^{n} \frac{\hat{N}_i^2}{n - k - 1}. \tag{6.2.9}$$

S_N^2 is the estimate of the variance of η, computed from the residuals $\hat{N}_i$, where

$$\hat{N}_i = e_i - [a_o + \sum_{j=1}^{k} a_j X_{ij}] \tag{6.2.10}$$

i.e., the N's are the residuals about the regression estimate of ϵ.

If the η's are assumed to be characterized by the normal distribution, then since the a's are linear combinations, they are also normal variates with expected value estimated by (6.2.7) and variances estimated by (6.2.8). Confidence intervals on the parameters $\sigma^2_{\alpha_i}$ and α_i can be obtained using the Chi-square and normal distributions as discussed in Section 5.2.

If the errors are not independent, then it can be shown that if this is not accounted for, the estimate of α will be biased. The techniques for handling correlated error come under regression analysis with "extended" or weighted least squares approximation. A detailed discussion of this topic is found in Reference 26.

6.3 ANALYSIS OF VARIANCE

Many of the previous sections show that error analysis is actually the analysis of the variation of measurements about some accepted standard. The problem involved in error analysis is to evaluate it in terms of explained variation (systematic error) and unexplained variation (random error). If one should obtain error estimates from several sources such as various tests, different instrumentation sites/systems, several runs, flights, times, aircraft, missiles, etc., he might desire to consolidate the results and design a technique to break the total variation into components which will explain the effects these varied sources have on the total variation. For example, the error models in Section 6.1 give several sources which will contribute to the total variance of the error. The questions are "how much do they contribute" and "is the contribution due to any one source significant?" If the answers to questions such as these can be obtained, we would then know which variables to use or delete in our regression analysis. The technique used to break variation in data into source components is called Analysis of Variance, which is highly dependent upon the design of experiments briefly mentioned in Section 6.2.

The sources of variation considered in analysis of variance are called variables or factors. The factors may be quantitative (such as a dynamic error model) or qualitative (such as different missiles or radars). Analysis of variance can correspond to several test designs. The most simple test design is the "completely randomized" or between-group and within-group design. In this case there is only one source of variation. For example, suppose that total error estimates y are made for a particular radar for k different tests from the same type of missile. The data would be grouped according to tests and the problem is to estimate the amount of variation (out of the total variation) between tests; the amount of variation within tests, and to test whether or not such variation is "significant." Such a test is called completely randomized because the grouping of the tests is accomplished in a random manner to eliminate any systematic trends which might exist from test to test. The mathematical model for such a design is

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \; j = 1, \ldots, n_i; \; i = 1, \ldots, k \tag{6.3.1}$$

where

y_{ij} = ith measurement from the jth group

n_i = number of observations in the ith group

μ = mean of population from which groups are sampled

α_i = deviation of mean of ith group from μ

ϵ_{ij} = unexplained variation in y_{ij}.

The α's are regarded as random variables with mean zero and variance σ_α^2. The term ϵ_{ij} is often called experimental error, and in the between-within experiment $\sigma_{\epsilon ij}^2$ would be the variance within the groups. Table 6.3.0 gives a typical between-within analysis of variance design (one-way analysis of variance) and illustrates how the parameters are estimated.

TABLE 6.3.0

ONE WAY ANALYSIS OF VARIANCE

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Squares	Expected Mean Square
Between groups	k-1	$\sum_i n_i(y_i - \bar{y})^2 = A$	A/k-1	$\sigma^2 + k\sigma_\alpha^2$
Within groups	N-k	$\sum_{i,j} (y_{ij} - \bar{y}_i)^2 = B$	B/N-k	σ^2
TOTAL	N-1	$\sum_{ij} (y_{ij} - \bar{y}_i)^2 = C$	$\frac{C}{N-1}$	

$$\bar{y}_i = \frac{1}{n_i} \sum_j y_{ij}, \quad N = \sum n_i$$

$$\bar{y} = \sum_i \frac{n_i \bar{y}_i)}{N}$$

$$k = (N^2 - \sum_i n_i^2)/[N(k-1)]$$

If the errors are not independent, the analysis of variance is not valid. Errors from different groups may reasonably be assumed uncorrelated, but errors within groups are often correlated, especially if the measurements within a group come from a time series. However, this can be offset by randomizing within each group. Thus, in order to be effective, the one-way analysis of variance should be completely randomized.

The analysis of variance usually has a "test of significance" associated with it. That is, we hypothesize that the means of each group are equal, then (under the assumption the groups have been completely randomized and are independent normal variates), test for the rejection of this hypothesis at a prescribed probability level. The most widely used test is the ratio of the variance from the source (between) to the variance of the experimental error. This forms a statistic which has the F probability distribution with (k-1) and (N-k) degrees of freedom. This is the same as saying

$$\text{Prob}\left[\frac{S_1^2}{S_2^2} = \frac{\text{variance between groups}}{\text{variance within groups}} < K\right] = \int_0^K F_{\substack{(N-k)\\(k-1)}} dF . \tag{6.3.2}$$

Suppose we select K such that $\text{Prob}(S_1^2/S_2^2 < k) = .95$, and then set up the hypothesis $H_0: \mu_1 = \mu_2 = \ldots = \mu_k = \mu$ (i.e., the means of each group are equal). From the analysis of variance we estimate F

$$\tilde{F} = \frac{S_1^2}{S_2^2} . \tag{6.3.3}$$

If $\tilde{F}$ is then compared to the theoretical value of F from the F distribution, and $\tilde{F} > F$ we reject H_0 and say the group means are significantly different. If $\tilde{F} < F$ we cannot reject H_0 and there is no significant difference.

When certain factors which are known are placed in the experimental design, additional constraints are added to the design. For example, we could select the p radars and k groups from the same type of test. We block the k groups and block the p radars in each of the k groups. This type is a randomized block design as shown below

TESTS

RADARS	1	2	...	k
1	y_{11}	y_{12}	⋮	y_{1k}
2	y_{21}	y_{22}	⋮	y_{2k}
⋮	⋮	⋮	⋮	⋮
P	y_{p1}	y_{p2}	...	y_{pk}

The analysis for the randomized block design is called a two-way analysis of variance, and the component variances measure the effects between tests and between radars with respect to the variance of the experimental error. If the cells in the design of Figure 6.3.0 contain r replicates each, then it becomes possible to estimate the effects of interaction between radars and tests.

The completely randomized design and the randomized block design are special cases of a more generalized design technique called factorial design. This technique handles several variables, each with several levels and each having replicates. This type of analysis could be used in conjunction with the error models (6.1.5) and (6.1.7). The error model for radar range error (Relation 6.1.5) would consist of 12 factors, each having n levels (corresponding to n observations), with one replicate. An example of an analysis of variance on a similar radar range error model is given in Table 6.3.1.

Table 6.3.1 is an example of an analysis which summarizes Sections 6.0, 6.1, 6.2, and 6.3, and needs some explanation. The coefficient of determination estimates the percentage of variation out of the total variation, which is explained by the regression analysis. The computed T value is the ratio of the regression coefficient to its standard error estimate. The partial correlation coefficient is an estimate of correlation between the two variables, keeping the effects of the other variables fixed. The multiple correlation coefficient is the correlation between the observed dependent variable and the regression estimate. The "proportion of variance cumulated" estimates the percent of the total variation due to each individual variable. Examination of the table shows that the fit explains only 49% of the total variations with 31.4% and 17.2% of this due to scale factor and timing. The T value

shows these two regression coefficients are significantly different from zero at the 95% confidence level. The total error estimate is RMS (ΔR) = 9.7 feet. The bias estimate is -7.54 feet, while the noise estimate is 4.63 feet. Since the ratio [(mean ΔR)/$\hat{\sigma}_{\Delta R}$)] is small it is difficult to obtain a large coefficient of determination (i.e., a large proportion of the total error is random or "unexplained"). However, the test for significance and T values indicates the regression fit is significant.

TABLE 6.3.1

ANALYSIS OF VARIANCE OF RADAR ERROR MODEL (EXAMPLE)

1. Error Model: $\Delta R_j = \alpha_0 + \alpha_1 \dot{R}_j + \alpha_3 \ddot{R}_j + N_j$
2. Coefficient of Determination: 0.4906
3. Multiple Correlation Coefficient: 0.7004
4. Standard Error of Estimate ($\hat{\sigma}_n$) = 4.63336
5. Intercept Estimate (a_o): -7.54538

ANALYSIS OF VARIANCE

Source of Variation	D.F.	Sum of Squares	Mean Squares	F Value
Due to regression	3	5396.28357	1798.76118	83.7879
Deviation about regression . .	261	5603.15576	21.46803	
TOTAL . .	264	10999.43933		

Variable	Mean	Std. Deviation	Reg. Coeff.	Std. Error of Reg.Coe.	Computed T Value	Partial Corr. Coe	Sum of Sq. Added	Prop. Var. Cum.
R	11829.42249	2329.27798	0.00132	0.00013	10.21585	0.53446	3455.36624	0.31414
$\dot{R}$	-0.34323	38.72200	0.07054	0.00746	9.45630	0.50516	1898.11476	0.17256
$\ddot{R}$	0.53988	0.27293	-1.54071	1.09114	-1.41202	-0.08707	42.80281	0.00389
ΔR	7.23463	6.45431						

7.0 THEORY OF ERROR PROPAGATION

In addition to estimating the error in the measurements from instrumentation systems, it is necessary to estimate the error in the final reduced data. This data usually consists of position, velocity, and acceleration information in some coordinate (usually rectangular) system. Thus, errors in range, azimuth, and elevation or direction cosines, film coordinates, range sums or differences must be propagated into rectangular coordinates. Furthermore, if velocity and acceleration are to be inferred from this positional data, the manner in which the errors propagate must be known. At least three factors are of importance here: (1) the manner in which observing sites are located with respect to each other (assuming that several are involved), (2) the errors in the measurement systems, and (3) the position of the object in space whose coordinates are to be determined.

Let x be a random variable with mean μ and variance σ_x^2. If $y = a_0 + a_1 x$, it can be shown that

$$\sigma_y^2 = a_1^2 \sigma_x^2 \tag{7.0.1}$$

Further, if $x_1, x_2, x_3, \ldots, x_n$ are independent random variables with variance σ_x^2, then the sum

$$y_j = \sum_{i=1}^{n} a_{ji} x_i, \quad j = 1,2,\ldots,N \tag{7.0.2}$$

has mean

$$\mu_{y_j} = \left(\sum_{i=1}^{n} a_{j_i} \right) \mu_x \tag{7.0.3}$$

and variance

$$\sigma_{y_j}^2 = \sum_{i=1}^{n} a_{j_i}^2 \sigma_x^2. \tag{7.0.4}$$

Further, the covariance of y_j, y_k, j, k = 1,2,..., N is

$$\sigma_{y_j y_k} = \left(\sum_{i=1}^{n} a_{j_i} a_{k_i} \right) \sigma_x^2 \tag{7.0.5}$$

If y = f(x) is some function, we can approximate f(x) in the neighborhood of $x = x_o$ by a first order Taylor series expansion. That is

$$y \doteq f(x_o) + \frac{df}{dx_o} (x - x_o) \tag{7.0.6}$$

In the same manner, using (7.0.1) we approximate the variance in y by the relation

$$\sigma_y^2 \doteq \left(\frac{df}{dx_o} \right)^2 \sigma_x^2 \, . \tag{7.0.7}$$

Now suppose $y = f(x_1,x_2)$, then the first order Taylor Series expansion in the neighborhood of $(x_1,x_2) = (x_1^o,x_2^o)$ is

$$y \doteq f(x_1^o,x_2^o) + \frac{\partial f}{\partial x_1} \Big|_{x_1^o} (x_1 - x_1^o) + \frac{\partial f}{\partial x_2} \Big|_{x_2^o} (x_2 - x_2^o), \tag{7.0.8}$$

and from (7.0.4) and (7.0.5) the form of σ_y^2 would be

$$\sigma_y^2 = \left(\frac{\partial f}{\partial x_1} \right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 \sigma_{x_2} + 2 \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \sigma_{x_1 x_2} \, . \tag{7.0.9}$$

In general if $y_i = y_i(x_1,x_2,\ldots,x_n)$, i = 1,2,...,m, then the Taylor Series expansion estimate of the covariance matrix of y is

$$\hat{\sigma}_y^2 = R S_x R^T \tag{7.0.10}$$

where

$$\hat{\sigma}_y^2 = \begin{bmatrix} \hat{\sigma}_{y_1}^2 & \hat{\sigma}_{y_1y_2} & \cdots & \hat{\sigma}_{y_1y_m} \\ \hat{\sigma}_{y_2y_1} & \hat{\sigma}_{y_2}^2 & \cdots & \hat{\sigma}_{y_2y_m} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\sigma}_{y_ny_1} & \hat{\sigma}_{y_ny_2} & \cdots & \hat{\sigma}_{y_m}^2 \end{bmatrix} \tag{7.0.11}$$

$$S_x = \begin{bmatrix} \hat{\sigma}_{x_1}^2 & \hat{\sigma}_{x_1x_2} & \cdots & \hat{\sigma}_{x_1x_n} \\ \hat{\sigma}_{x_2x_1} & \hat{\sigma}_{x_2}^2 & \cdots & \hat{\sigma}_{x_2x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\sigma}_{x_nx_1} & \hat{\sigma}_{x_nx_2} & \cdots & \hat{\sigma}_{x_n}^2 \end{bmatrix} \tag{7.0.12}$$

$$R = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \tag{7.0.13}$$

and

R^T is the transpose of R

If the x_i's are independent random variables then S_x is a diagonal matrix, i.e., the covariances are all zero. In this case

$$\sigma_{y_iy_j} = \frac{\partial y_i}{\partial x_1} \frac{\partial y_j}{\partial x_1} \sigma_x^2, + \left(\frac{\partial y_i}{\partial x_2}\right) \sigma_{x_2}^2 + \ldots + \left(\frac{\partial y_i}{\partial x_n}\right) \sigma_{x_n}^2 \tag{7.0.14}$$

The relation $\sigma_y^2 = RS_xR^T$ is usually called the covariance equation and is classically referred to as the Gaussian law of error propagation. This technique is based directly upon approximating the change in the dependent variable by the differential, i.e., the linear portion of a Taylor Series expansion. This leads one to wonder how much error is involved in the variance estimates. Actual examples using simulation techniques have shown the method to be very good and, as John Tukey is quoted as saying, "The most important conclusion is that the classical propagation formula is much better than seems to be realized. Examples indicate that it is quite likely to suffice for most work."

7.1 PROPAGATION OF ERRORS INTO POSITION

Since most of the measured quantities are angle, direction cosines, film coordinates, range sums, and range differences, the effects of errors in these observations are present in rectangular coordinates derived from these quantities. While it is most convenient to consider the errors in the actual measurements of the instrumentation system when discussing its accuracy, these errors do not necessarily give the desired information. The relation between errors in range, azimuth, and elevation and errors in derived position data depends upon three factors: (1) the location and geometry of the instrumentation sites, (2) the errors in the measurements of the system, and (3) the position of the point in space that is to be determined.

Methods can be developed and programmed which give estimates of the errors in rectangular coordinates when items (1), (2), (3) in the above paragraph are taken into consideration. The end product of these programs tests estimates of errors in rectangular coordinates, and is referred to as geometric dilution of precision (GDOP). For examining the capabilities of given instrumentation on a particular test, significant estimates of the rectangular coordinate accuracies can be obtained by using the nominal trajectory for the test, the actual instrument sites to be used, and the best estimates of the errors of measurement for the particular systems. Further information on GDOP may be obtained from the Bibliography.

Although GDOP is widely used, the more sophisticated concept of an ellipsoid of error is gaining acceptance as spatial measurements are made with higher orders of precision. Whereas GDOP utilizes only information from the diagonal terms ($\sigma_X^2, \sigma_Y^2, \sigma_Z^2$) of the variance-covariance matrix, the ellipsoid of error utilizes all of the information which can be extracted from the entire matrix. If an ellipsoid of error is used, confidence regions can be set up about each spatial position point and the orientation of the axes of the ellipsoid can be obtained. This is discussed in more detail in Section 8.0.

The basic problem in error propagation is to determine the effect of the random errors of measurement in the particular instruments on the rectangular coordinate data. Suppose, for example, that the quantities, α, β, γ, are measured at time, t, with errors $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$. If there is a known mathematical relation between X, Y, Z and α, β, γ:

$$\begin{aligned} X &= f_1(\alpha,\beta,\gamma), \\ Y &= f_2(\alpha,\beta,\gamma), \\ Z &= f_3(\alpha,\beta,\gamma), \end{aligned} \tag{7.1.0}$$

then the errors in X, Y, and Z produced by $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$ can be approximated by

$$\begin{aligned} \Delta X &= \frac{\partial X}{\partial \alpha}\Delta\alpha + \frac{\partial X}{\partial \beta}\Delta\beta + \frac{\partial X}{\partial \gamma}\Delta\gamma, \\ \Delta Y &= \frac{\partial Y}{\partial \alpha}\Delta\alpha + \frac{\partial Y}{\partial \beta}\Delta\beta + \frac{\partial Y}{\partial \gamma}\Delta\gamma, \\ \Delta Z &= \frac{\partial Z}{\partial \alpha}\Delta\alpha + \frac{\partial Z}{\partial \beta}\Delta\beta + \frac{\partial Z}{\partial \gamma}\Delta\gamma, \end{aligned} \tag{7.1.1}$$

where the partial derivatives are evaluated using f_1, f_2, f_3 and the measured values of α, β, γ. This type of estimation of the errors is also satisfactory when discussing the effects of systematic errors if these errors are relatively small with respect to the magnitudes of X, Y and Z.

A discussion for other systems involving multiple stations and least square solutions is based upon the same ideas. However, they are complicated by the fact that more variables are involved and more complex mathematical manipulation must be carried out. For simplicity, methods are illustrated here for radar.

The relations between the measured quantities, azimuth (A), elevation (E), and range (R) and the space position (X,Y,Z) in left handed coordinates are:

$$X = R \cos A \cos E$$

$$Y = R \sin A \cos E \tag{7.1.2}$$

$$Z = R \sin E.$$

Errors may exist singly in one of the measurement R, A, E or simultaneously in R, A, E. Using the above relations and simultaneous errors in R, A, E, the first order effects of these errors upon X, Y, Z are given by:

$$\begin{aligned} \Delta X &\doteq \Delta R \cos E \cos A - R\,\Delta A \cos E \sin A - R\,\Delta E \sin E \cos A \\ \Delta Y &\doteq \Delta R \cos E \sin A + R\,\Delta A \cos E \cos A - R\,\Delta E \sin E \sin A \\ \Delta Z &\doteq \Delta R \sin E + R\,\Delta E \cos E \end{aligned} \tag{7.1.3}$$

Using the covariance equation (7.0.10), we estimate the elements of the variance-covariance matrix σ for the radar assuming the errors in R, A, E are uncorrelated. The estimates are

$$\sigma_X^2 = \left(\frac{\partial X}{\partial R}\right)^2 \sigma_R^2 + \left(\frac{\partial X}{\partial A}\right)^2 \sigma_A^2 + \left(\frac{\partial X}{\partial E}\right)^2 \sigma_E^2$$

$$= \sigma_R^2 \cos^2 A \cos^2 E + \sigma_E^2 R^2 \cos^2 A \sin^2 E + \sigma_A^2 R^2 \sin^2 A \cos^2 E,$$

$$\sigma_Y^2 = \left(\frac{\partial Y}{\partial R}\right)^2 \sigma_R^2 + \left(\frac{\partial Y}{\partial A}\right)^2 \sigma_A^2 + \left(\frac{\partial Y}{\partial E}\right)^2 \sigma_E^2$$

$$= \sigma_R^2 \sin^2 A \cos^2 E + \sigma_A^2 R^2 \cos^2 A \cos^2 E + \sigma_E^2 R^2 \sin^2 A \sin^2 E,$$

$$\sigma_Z^2 = \left(\frac{\partial Z}{\partial R}\right)^2 \sigma_R^2 + \left(\frac{\partial Z}{\partial A}\right)^2 \sigma_A^2 + \left(\frac{\partial Z}{\partial E}\right)^2 \sigma_E^2$$

$$= \sigma^2 \sin^2 E + \sigma_E^2 R^2 \cos^2 E,$$

$$\sigma_{xy} = \frac{\partial X}{\partial R}\frac{\partial Y}{\partial R}\sigma_R^2 + \frac{\partial X}{\partial A}\frac{\partial Y}{\partial A}\sigma_A^2 + \frac{\partial X}{\partial E}\frac{\partial Y}{\partial E}\sigma_E^2 \qquad (7.1.4)$$

$$= (\cos A \sin A \cos^2 E)\,\sigma_R^2 - (R^2 \cos E^2 \sin A \cos A)\,\sigma_A^2$$

$$+ (R^2 \cos A \sin^2 E \sin A)\,\sigma_E^2,$$

$$\sigma_{xz} = \frac{\partial X}{\partial R}\frac{\partial Z}{\partial R}\sigma_R^2 + \frac{\partial X}{\partial A}\frac{\partial Z}{\partial A}\sigma_A^2 + \frac{\partial X}{\partial E}\frac{\partial Z}{\partial E}\sigma_E^2$$

$$= (\cos A \cos E \sin E)\,\sigma_R^2 - (R^2 \cos A \sin E \cos E)\,\sigma_E^2,$$

$$\sigma_{yz} = \frac{\partial Y}{\partial R}\frac{\partial Z}{\partial R}\sigma_R^2 + \frac{\partial Y}{\partial A}\frac{\partial Z}{\partial A}\sigma_A^2 + \frac{\partial Y}{\partial E}\frac{\partial Z}{\partial E}\sigma_E^2$$

$$= (\sin A \cos E \sin E)\,\sigma_R^2 - (R^2 \sin A \sin E \cos E)\,\sigma_E^2.$$

The variances and covariances correspond to the error distribution which has been propagated into a left handed coordinate system tangent to the spheroid with the radar at the origin of the system.

In many instances, it is desired to propagate the error into a coordinate system with a different origin and orientation. For example, it is often desired to propagate the error into a coordinate system whose origin is a launching pad and whose +X axis is down the launch azimuth. In this case the R A E data is first transformed into a local tangent plane system, (X_{TP},$Y_{TP}$$Z_{TP}$). Based on the geodetic position (ϕ,λ) of the radar, the tangent plane coordinates are transformed so that each axis is parallel to a geocentric (earth centered) system, translated to geocentric origin, translated to new origin (launch pad), then the axes are rotated into the tangent plane system with the +X axis parallel to the desired orientation. This process is accomplished by the following matrix equation

$$[P]_{\alpha}^{T} \left[[R] \begin{bmatrix} X_{TP} \\ Y_{TP} \\ Z_{TP} \end{bmatrix} + [G_R] - [G_P] \right] = \begin{bmatrix} X'_{TP} \\ Y'_{TP} \\ Z'_{TP} \end{bmatrix} \qquad (7.1.5)$$

where

[R] = matrix which rotates radar tangent plane into geocentric system

$[G_R]$ and $[G_P]$ are geocentric coordinates of radar and launch pad

$[P]^T$ = Matrix which rotates axes parallel to local tangent plane so translation from earth's center can be obtained. [P] corresponds to [R] only is for the launch pad.

α = orientation azimuth of transformed system

$\begin{bmatrix} X'_{TP} \\ Y'_{TP} \\ Z'_{TP} \end{bmatrix}_{\alpha}$ = local tangent plane coordinates with origin at launch pad P with +X axis oriented parallel to α degrees

The matrices [R] and [P] are linear transformations whose elements are constants which are functions of α, ϕ and λ. Since the transformations are linear, the propagated estimate for the covariance matrix in the new coordinate system is

$$\hat{\sigma}_P = A \, \hat{\sigma}_{TP} \, A^T \qquad (7.1.6)$$

where $\hat{\sigma}_{TP}$ = covariance matrix with respect to radar tangent plane

$A = [P]^T [R]$

It is pointed out that the error propagation techniques can propagate from X,Y,Z back to the original observations of R,A,E. In order to do this the geometry (X,Y,Z) and the covariance matrix must be known. For example, for

$$R = f(X,Y,Z)$$

then

$$\sigma_R^2 = \left(\frac{\partial R}{\partial X}\right)^2 \sigma_X^2 + \left(\frac{\partial R}{\partial Y}\right)^2 \sigma_Y^2 + \left(\frac{\partial R}{\partial Z}\right) \sigma_Z^2$$

$$+ 2 \left(\frac{\partial R}{\partial X}\right)\left(\frac{\partial R}{\partial Y}\right) \sigma_{XY} + 2 \left(\frac{\partial R}{\partial X}\right)\left(\frac{\partial R}{\partial Z}\right) \sigma_{XZ} + 2 \left(\frac{\partial R}{\partial Y}\right)\left(\frac{\partial R}{\partial Z}\right) \sigma_{YZ}. \quad (7.1.7)$$

Should it be necessary to differentiate the position to acquire velocity, it is assumed that any low-frequency systematic error is virtually eliminated in the process. It is noted, however, that cyclic errors may not be disposed of in this manner.

It is noted that, under certain conditions, the effects of total errors can be propagated into spatial position error. If the variance of the systematic error corresponding to the same point in space is estimated over several tests, then the effects of this can be propagated into spatial position error. In this case the variance of the total error input for propagation would be

$$\sigma^2_{TOTAL} = \sigma^2_{SYSTEMATIC} + \sigma^2_{RANDOM} \quad (7.1.8)$$

and the propagated error would have a variance-covariance matrix corresponding to total spatial position error.

7.2 PROPAGATION OF ERRORS INTO VELOCITY AND ACCELERATION

In general, position and range rate data are the basic measurements received by instrumentation systems which are external to a vehicle in flight. The basic trajectory-related measurements from on board systems are acceleration components parallel to the missile coordinate system. In the first case, velocity and acceleration data are obtained by numerically estimating the first and second derivatives of the position data. In the latter, velocity and position are obtained by successive integration with respect to time.

Very low frequency systematic errors in position data may be regarded as constant over short spans of time. In this instance a measured variable can be represented as a function of time by

$$V = F(t) + B \tag{7.2.1}$$

where B is the bias error. Upon numerical differentiation with respect to time we obtain

$$\dot{V} = F'(t) \tag{7.2.2}$$

which is independent of B. The significance of this fact is that a differentiation derived from a measured function with low frequency bias error is virtually free of this bias. However, higher frequency cyclic error and random error will persist and must be considered. The amount of random error for a particular instant of time is, of course, unknown and cannot be predicted as a function of time, but under the assumption that it is a stationary process (at least during short spans of data), its statistical properties remain constant.

Inertial guidance systems currently in use in many missile programs provide examples of acceleration measuring devices. The functions these instruments perform are varied and complicated. Inertial guidance systems are discussed in more detail in Section 10.0. In this section we are concerned primarily with errors propagated due to numerical differentiation of position data with respect to time.

If "raw" velocities in space position data were computed by successive differences, the "average" velocity estimate over the time increment would be

$$\dot{X} \doteq \Delta X_{\frac{2i+1}{2}} = \frac{1}{\Delta t}(X_i - X_{i-1}). \tag{7.2.3}$$

The X_i's are random variables and assumed to be independent with mean μ and variance σ_X^2. Then, using the propagation techniques of Section 7.0, the variance in ΔX is

$$\text{Var}\left(\frac{\Delta X}{\Delta t}\right) = \frac{\sigma_{X_i}^2 + \sigma_{X_{i-1}}^2}{\Delta t^2} = \frac{2\sigma_X^2}{\Delta t^2}. \tag{7.2.4}$$

From (7.2.4) it is obvious that the random variable ΔX contains more random error than X. For example, if σ_X = 2 feet and Δt = 0.1 second, then the error in raw velocity would be $\pm 20\sqrt{2}$ or ± 28.28 feet per second. In the case of "raw" acceleration it is much worse, and for σ_X = 2 feet, $\sigma_{\ddot{X}} = \pm 500$ feet per second^2! These examples were given to show that filtering/smoothing techniques are needed to achieve reasonable $\dot{X}$ and $\ddot{X}$ estimates when derived from numerically differentiating position data.

Most methods presently used filter the position data as a function of time and then numerically differentiate the filtered data. The most common type of filter used is a kth degree polynomial which is fitted by some criterion (usually least squares) to a span of m points. The first and second derivatives of the polynomial are evaluated at an appropriate time, usually the midpoint, since this gives velocity and acceleration data with smaller random errors than end point estimates. The point of evaluation which gives minimum random error is between the end and center points. The moving arc technique of fitting the polynomial of kth degree of the data, X_2 to X_{m+1} and t_2 to t_{m+1}, and evaluating first and second derivatives is used. This is continued until all desired velocity and acceleration components are computed. In the so-called simple differencing method, the degree of the polynomial is 2. If the data is equally spaced in time, i.e.,

$$t_j = t_0 + (\Delta t)\, j, \tag{7.2.5}$$

then the polynomial filters for velocity and acceleration will consist of weights b_j and c_j and the velocity and acceleration estimates are made from the following linear combinations:

$$\frac{dX_i}{dt} \doteq \frac{1}{\Delta t} \sum_{j=1}^{m} b_j\, X_{i-j+d} \tag{7.2.6}$$

and

$$\frac{d^2X_i}{dt^2} \doteq \frac{1}{(\Delta t)^2} \sum_{j=1}^{m} c_j\, X_{i-j+d} \tag{7.2.7}$$

respectively, where d is related to the delay (i.e., if m = 2n+1 and evaluation is at the end point then d = m). The b_j and c_j depend on the degree k of the polynomial, the span of points, m, used in the moving average, and the number d. The b_j and c_j are fixed constants and are precomputed. Using formulas of the type above and assuming that a polynomial of kth degree approximates the data to a sufficient degree of accuracy over the span of m points, and that the errors in successive values of X_j are uncorrelated, the relation between the random errors in position data and the random errors in velocity and acceleration data can be expressed in the form

$$\hat{\sigma}_{\dot{X}} = \frac{\hat{\sigma}_X}{\Delta t} \left(\sum_{j=1}^{m} b_j^2 \right)^{1/2} \tag{7.2.8}$$

and

$$\hat{\sigma}_{\ddot{X}} = \frac{\hat{\sigma}_X}{(\Delta t)^2}\left(\sum_{j=1}^{m} c_j^2\right)^{\frac{1}{2}} \tag{7.2.9}$$

Similar relations can be obtained if the errors are correlated and the form of the correlation is known. In the case of midpoint smoothing the equations for $\sigma_{\dot{X}}$ and $\sigma_{\ddot{X}}$ become

$$\hat{\sigma}_{\dot{X}} = \frac{\hat{\sigma}_X}{\Delta t}\sqrt{\frac{12}{m(m^2-1)}} \simeq \frac{\sigma_X}{\Delta tm}\frac{\sqrt{12}}{\sqrt{m}}, \tag{7.2.10}$$

for second degree polynomial, and

$$\hat{\sigma}_{\ddot{X}} = \frac{\sigma_X}{(\Delta t)^2}\sqrt{\frac{720}{m(m^2-1)(m^2-4)}} \simeq \frac{\sigma_X}{(\Delta tm)^2}\frac{\sqrt{720}}{\sqrt{m}} \tag{7.2.11}$$

for a third degree polynomial. It is evident from the approximations that if the sampling rate is stepped up and the smoothing time span (Δtm) held constant, the velocity and acceleration errors are reduced approximately by a factor of $m^{-\frac{1}{2}}$. Truncation errors limit the extent to which this can be applied. If smoothing is performed at some point other than the midpoint the same general argument applies.

In the discussion of the effects of random errors on velocity and acceleration, it was pointed out that the method presented was valid only if the errors in successive values of a given coordinate are uncorrelated. This assumption is not true for a systematic error by definition. Hence, different techniques are required for determination of the effects of systematic errors.

In the case of single station systems, the errors in velocity and acceleration in Cartesian coordinates may be found by differentiating the XYZ equations with respect to time and utilizing the Gaussian approximation.

$$\sigma_{f_i}^2 = \sum\left(\frac{\partial f_i}{\partial \alpha_j}\right)^2 \sigma_{\alpha_j}^2 \tag{7.2.12}$$

where f_i = X, Y, Z, $\dot{X}$, $\dot{Y}$, $\dot{Z}$ and α_j = R, A, E, $\dot{R}$, $\dot{A}$, $\dot{E}$ for radar. For example, the variances of the velocity errors for radar become

$$\sigma_{\dot{X}}^2 = (-\dot{E} \sin E \cos A - \dot{A} \cos E \sin A)^2 \sigma_R^2$$

$$+ (-\dot{R} \sin E \cos A - \dot{E} R \cos E \cos A + \dot{A} R \sin E \sin A)^2 \sigma_E^2 \quad (7.2.13)$$

$$+ (-\dot{R} \cos E \sin A + \dot{E} R \sin E \sin A - \dot{A} R \cos A \cos E)^2 \sigma_A^2$$

$$+ (\cos A \cos E)^2 \sigma_{\dot{R}}^2 + (-R \cos A \sin E)^2 \sigma_{\dot{E}}^2 + (-R \sin A \cos E)^2 \sigma_{\dot{A}}^2$$

$$\sigma_{\dot{Y}}^2 = (-\dot{E} \sin E \sin A + \dot{A} \cos E \cos A)^2 \sigma_R^2$$

$$+ (\dot{R} \cos E \cos A - \dot{E} R \sin E \cos A - \dot{A} R \cos E \sin A)^2 \sigma_A^2 \quad (7.2.14)$$

$$+ (-\dot{R} \sin E \sin A - \dot{E} R \cos E \sin A - \dot{A} R \sin E \cos A)^2 \sigma_E^2$$

$$+ (\sin A \cos E)^2 \sigma_{\dot{R}}^2 + (-R \sin A \sin E)^2 \sigma_{\dot{E}}^2 + (R \cos E \cos A)^2 \sigma_{\dot{A}}^2$$

$$\sigma_{\dot{Z}}^2 = (\dot{E} \cos E)^2 \sigma_R^2 + (\dot{R} \cos E - \dot{E} R \sin E)^2 \sigma_E^2 \quad (7.2.15)$$

$$+ (\sin E)^2 \sigma_{\dot{R}}^2 + (R \cos E)^2 \sigma_{\dot{E}}^2$$

The expressions for the covariance terms $\sigma_{\dot{x}\ddot{y}}$, $\sigma_{\dot{x}\ddot{z}}$, $\sigma_{\dot{y}\ddot{z}}$ may be written according to the following expression for $\sigma_{\dot{x}\ddot{y}}$.

$$\sigma_{\dot{x}\ddot{y}} = \sigma_{\ddot{y}\dot{x}} = \frac{\partial \dot{X}}{\partial R}\frac{\partial \ddot{Y}}{\partial R}\sigma_R^2 + \frac{\partial \dot{X}}{\partial A}\frac{\partial \ddot{Y}}{\partial A}\sigma_A^2 + \frac{\partial \dot{X}}{\partial E}\frac{\partial \ddot{Y}}{\partial E}\sigma_E^2 + \frac{\partial \dot{X}}{\partial \dot{R}}\frac{\partial \ddot{Y}}{\partial \dot{R}}\sigma_{\dot{R}}^2 + \frac{\partial \dot{X}}{\partial \dot{A}}\frac{\partial \ddot{Y}}{\partial \dot{A}}\sigma_{\dot{A}}^2 + \frac{\partial \dot{X}}{\partial \dot{E}}\frac{\partial \ddot{Y}}{\partial \dot{E}}\sigma_{\dot{E}}^2 \quad . \tag{7.2.16}$$

In a similar manner, a partitioned variance-covariance matrix can be estimated giving all position, velocity, acceleration error variances and covariances. The matrix would be

$$\bar{\sigma} = \left|\begin{array}{ccc:ccc:ccc} \sigma_X^2 & \sigma_{xy} & \sigma_{xz} & \sigma_{x\dot{x}} & \sigma_{x\dot{y}} & \sigma_{x\dot{z}} & \sigma_{x\ddot{x}} & \sigma_{x\ddot{y}} & \sigma_{x\ddot{z}} \\ \sigma_{xy} & \sigma_Y^2 & \sigma_{yz} & \sigma_{y\dot{x}} & \sigma_{y\dot{y}} & \sigma_{y\dot{z}} & \sigma_{y\ddot{x}} & \sigma_{y\ddot{y}} & \sigma_{y\ddot{z}} \\ \sigma_{xz} & \sigma_{yz} & \sigma_Z^2 & \sigma_{z\dot{x}} & \sigma_{z\dot{y}} & \sigma_{z\dot{z}} & \sigma_{z\ddot{x}} & \sigma_{z\ddot{y}} & \sigma_{z\ddot{z}} \\ \hdashline \sigma_{\dot{x}x} & \sigma_{\dot{x}y} & \sigma_{\dot{x}z} & \sigma_{\dot{X}}^2 & \sigma_{\dot{x}\dot{y}} & \sigma_{\dot{x}\dot{z}} & \sigma_{\dot{x}\ddot{x}} & \sigma_{\dot{x}\ddot{y}} & \sigma_{\dot{x}\ddot{z}} \\ \sigma_{\dot{y}x} & \sigma_{\dot{y}y} & \sigma_{\dot{y}z} & \sigma_{\dot{x}\dot{y}} & \sigma_{\dot{Y}}^2 & \sigma_{\dot{y}\dot{z}} & \sigma_{\dot{y}\ddot{x}} & \sigma_{\dot{y}\ddot{y}} & \sigma_{\dot{y}\ddot{z}} \\ \sigma_{\dot{z}x} & \sigma_{\dot{z}y} & \sigma_{\dot{z}z} & \sigma_{\dot{x}\dot{z}} & \sigma_{\dot{y}\dot{z}} & \sigma_{\dot{Z}}^2 & \sigma_{\dot{z}\ddot{x}} & \sigma_{\dot{z}\ddot{y}} & \sigma_{\dot{z}\ddot{z}} \\ \hdashline \sigma_{\ddot{x}x} & \sigma_{\ddot{x}y} & \sigma_{\ddot{x}z} & \sigma_{\ddot{x}\dot{x}} & \sigma_{\ddot{x}\dot{y}} & \sigma_{\ddot{x}\dot{z}} & \sigma_{\ddot{X}}^2 & \sigma_{\ddot{x}\ddot{y}} & \sigma_{\ddot{x}\ddot{z}} \\ \sigma_{\ddot{y}x} & \sigma_{\ddot{y}y} & \sigma_{\ddot{y}z} & \sigma_{\ddot{y}\dot{x}} & \sigma_{\ddot{y}\dot{y}} & \sigma_{\ddot{y}\dot{z}} & \sigma_{\ddot{x}\ddot{y}} & \sigma_{\ddot{Y}}^2 & \sigma_{\ddot{y}\ddot{z}} \\ \sigma_{\ddot{z}x} & \sigma_{\ddot{z}y} & \sigma_{\ddot{z}z} & \sigma_{\ddot{z}\dot{x}} & \sigma_{\ddot{z}\dot{y}} & \sigma_{\ddot{z}\dot{z}} & \sigma_{\ddot{x}\ddot{z}} & \sigma_{\ddot{y}\ddot{z}} & \sigma_{\ddot{Z}}^2 \end{array}\right| \tag{7.2.17}$$

In the case of multiple station systems requiring least squares solutions for data reduction, the variances cannot be readily determined analytically as for radar. The same reasonable assumption is made, however, that the variances of the instrument errors represents a normal distribution at the same point in the trajectory over many tests. With this the Gaussian approximation is again utilized, except that the partial derivatives are replaced by increments, i.e.,

$$\frac{\partial f_i}{\partial \alpha_j} \approx \frac{\Delta f_i}{\Delta \alpha_j}$$

The ratio of $\Delta f_i/\Delta \alpha_j$ is determined by numerically computing Δf_i with a predetermined value of $\Delta \alpha_j$ with all other j values of the instrument variables held constant.

8.0 ERROR TOLERANCE REGIONS

Confidence and tolerance intervals on the point estimates of error parameters were discussed in Section 5.2. The next points in question are confidence regions on two- and three-dimensional spatial position data. In general, the type of tolerance regions used consist of ellipses in the two-dimensional case and ellipsoids in the three-dimensional cases. (Special cases of these are circles and spheres.)

In obtaining any type of confidence/tolerance region one must first make an assumption concerning the underlying probability distribution function, and then know or have a good estimate on the parameters which characterize the distribution. The usual assumption is that the underlying distribution is approximately Gaussian which means that all nth dimension distributions can be characterized in terms of the first and second moments of the distribution (i.e., the means and variances). Thus, the basic tool for the confidence region is the variance or variance-covariance matrix. The probability density function for the Gaussian distribution in three dimensions is

$$f(X,Y,Z) = \frac{1}{(2\pi)^{3/2}\,|\sigma|^{1/2}}\, e^{-1/2\; Q^{-1}(X,Y,Z)}$$

where

$|\sigma|$ = determinant of covariance matrix σ

$Q^{-1}(X,Y,Z)$ = quadratic form of the inverse covariance matrix σ^{-1}

The quadratic form of the matrix is

$$(X, Y, Z)\,[\sigma^{-1}]\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = X^2\sigma_x^{2\,\prime} + Y^2\sigma_y^{2\,\prime} + Z^2\sigma_z^{2\,\prime} + 2XY\sigma_{xy}^{\prime} + 2XZ\sigma_{xz}^{\prime} + 2YZ\sigma_{yz}^{\prime} \qquad (8.0.1)$$

where the prime designates the elements of σ^{-1}. The above relation is the general equation of an ellipsoid and it can be transformed into standard form

$$\frac{\hat{X}^2}{\sigma_{\hat{x}}^{2\prime}} + \frac{\hat{X}^2}{\sigma_{\hat{y}}^{2\prime}} + \frac{\hat{Z}^2}{\sigma_{\hat{z}}^{2\prime}} = 1 \qquad (8.0.2)$$

by rotating the axes of the general form equation parallel to the $(\hat{X},\hat{Y},\hat{Z})$ coordinate system. It can be shown that this process is equivalent to diagonalizing the inverse covariance matrix which, in turn, is equivalent to finding the eigenvalues of σ^{-1}.

The diagonal matrix D of σ^{-1} is the matrix of eigenvalues

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad (8.0.3)$$

and its quadratic form is

$$[\hat{X},\ \hat{Y},\ \hat{Z}]\, D \begin{pmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{pmatrix} = \lambda_1 \hat{X}^2 + \lambda^2 \hat{Y}^2 + \lambda_3 \hat{Z}^2 = C^2$$

or

$$\frac{\hat{X}^2}{C^2/\lambda_1} + \frac{\hat{Y}^2}{C^2/\lambda_2} + \frac{\hat{Z}^2}{C^2/\lambda_3} = 1. \tag{8.0.4}$$

The variances of the standard trivariate or ellipsoidal normal distribution are

$$\sigma^2_{\hat{x}} = C^2/\lambda_1,\ \sigma^2_{\hat{y}} = C^2/\lambda_2,\ \sigma^2_{\hat{z}} = C^2/\lambda_3 \tag{8.0.5}$$

It is noted that if the errors in X, Y and Z are independent, then the general equation of the ellipsoid has no cross product terms and corresponds to a matrix σ^{-1}, which is already a diagonal matrix. If the errors are independent and $\sigma^2_x = \sigma^2_y = \sigma^2_z$, the distribution is the spherical normal distribution (3 dimensions) or the circular normal when two axes are equal and the third is zero.

When the errors in X, Y and Z are correlated, then σ has off-diagonal terms and the corresponding error ellipsoid is not in standard form, but is skewed in space. The eigenvalues of σ^{-1} provide the length of the axes and the eigenvectors of σ^{-1} will tell the orientation of each of the axes with respect to the coordinate system $(\dot{X},\dot{Y},\dot{Z})$.

8.1 TOLERANCE ELLIPSOIDS

Once an error ellipsoid has been obtained, the question of how much of the spatial error distribution is contained in the ellipsoid is encountered. Since the errors are assumed to be normal, it can be shown that the axes of the standard ellipsoid equation have the Chi-square distribution with three degrees of freedom. With three degrees of freedom, the probability, p, that a point will be in an ellipsoid with constant term C^2 is

$$p = \int_0^{C^2} \chi_3^2 \, d\chi^2 \tag{8.1.0}$$

The value of C^2 corresponding to p can be obtained from any Chi-square table. The following table gives values of C versus p.

p	C
.95	2.79
.90	2.50
.80	2.15
.70	1.91
.50	1.54
.20	1.00

TABLE 1

Thus, for a 90% error tolerance ellipse, C = 2.50 so that the axes would be

$$\sigma_{\hat{x}} = \frac{2.5}{\sqrt{\lambda_1}}, \quad \sigma_{\hat{y}} = \frac{2.5}{\sqrt{\lambda_2}}, \quad \sigma_{\hat{z}} = \frac{2.5}{\sqrt{\lambda_3}} \tag{8.1.1}$$

The values of C = 1.538 and C = 4.0 are the values of C corresponding to "probable" ellipsoid of error and ellipsoidal near-certainty error, respectively.

In the two-dimensional case, the same procedure is used except that σ^{-1} is a 2x2 matrix and the standard equation of the ellipse is Chi-square with two degrees of freedom. A table of p versus C is given below.

p	C
.99	3.37
0.95	2.45
0.90	2.146
0.865	2.00
0.50	1.177
0.394	1.00

TABLE 2

When C = 1.774 we have a tolerance ellipse called the probable ellipse of error. When the axes are equal, the tolerance region corresponding to C = 1.774 is called "Circular Probable Error" (CPE).

In actual practice $\sigma_x^2 \neq \sigma_y^2 \neq \sigma_z^2$ and the errors will not be independent, so it is usually more valid to work with ellipsoids and ellipses rather than circles or spheres. There are certain instances, however, when it is desired to obtain a circular or spherical tolerance region from an ellipse or ellipsoid. That is, it is desired to find the radius of a circle (or sphere) which contains the same number of points as the tolerance ellipse (ellipsoid) with axes $C/\sqrt{\lambda_1}$, $C/\sqrt{\lambda_2}$. To accomplish this, we make the change of variables letting $\hat{x} = r \cos \theta$, $\hat{y} = r \sin \theta$ then integrating the bivariate elliptical normal distribution with respect to r and θ we find p as a function of

$$A = \text{Max}\left(\frac{\sigma_{\hat{x}}}{\sigma_{\hat{y}}}, \frac{\sigma_{\hat{y}}}{\sigma_{\hat{x}}}\right) \quad \text{and} \quad B = \frac{R_p}{\sqrt{\sigma_{\hat{x}}\sigma_{\hat{y}}}}$$

where R_p is the radius corresponding to p. The following table gives A and B versus p = 0.90.

A	B
1.0	2.1460
1.5	2.2501
2.0	2.4565
2.5	2.6865
3.0	2.9118
4.0	3.3290
5.0	3.7058
10.0	5.2111
15.0	6.3756
20.0	7.3594
50.0	11.631
100.0	16.449

TABLE 3

For example, if the maximum ratio of $\sigma_{\dot{x}}$ to $\sigma_{\dot{y}}$ is A = 2.0 then the radius of a circle containing 90% of the error distribution would be $R_{.90} = 2.4565\sqrt{\sigma_{\dot{x}}\sigma_{\dot{y}}}$.

The radius R_p of a sphere containing p% of an error distribution can be obtained from a probability ellipsoid in a similar but more complicated manner, and will not be discussed in this report (See Reference 29 for details).

9.0 DIGITAL FILTERING/SMOOTHING OF DATA

As stated in Section 2.0, the measured or raw data contains signal with error superimposed on it. The basic reason for using a digital filter is to separate or suppress the noise error from the signal. There are many different filtering techniques used at the various ranges but, in general, a digital filter is evaluated by the following criteria:

1) How effectively the noise components of the input data are attenuated or removed.

2) How much distortion in the signal.

3) How effectively the filter recovers and smooths the first, second, and higher order derivatives.

4) Amount of serial correlation in the data.

5) Amount of computing time required.

It would be desirable to select a filter which would give the best of each item above, but unfortunately this is not usually the case. Often a filter may be designed to accomplish 1, 2, or 3 in some "best" manner, but usually Item 5 is then sacrificed.

In general, most filters used are linear operators on the raw data. If $X_1 \ldots, X_n$ is a set of raw data, the filtered data is usually expressed as a linear combination of subsets or data spans of the set. For example, if the filter gives a point estimate corresponding to the center time point of a span consisting of 2N+1 points then the expression for the output of the filter would be

$$\hat{X}_t = \sum_{i=-N}^{i=+N} W_i X_{t+i} \qquad (9.0.1)$$

where the W_i are filter weights. These weights are constrained so that

$$\sum W_i = 1 \qquad (9.0.2)$$

This is necessary since the signals occuring in missile trajectory work usually have a very large low-frequency trend, and the response of the filter at these low frequencies should be exactly one (i.e., no distortion or biasing at zero frequency).

If the filter is an end point estimation or "predictor" filter, then the weighted sum would consist of k previous points.

Many of the digital filters being used are of the general-purpose type. The parameters which define the weights of such a filter are adjusted according to the sampling rate of the input data and some estimate as to the frequency composition of the signal. Some types of general-purpose filters are

1) Straight moving average

2) Classical least squares polynomial fit

3) Constrained least squares polynomial fit

4) Variable span

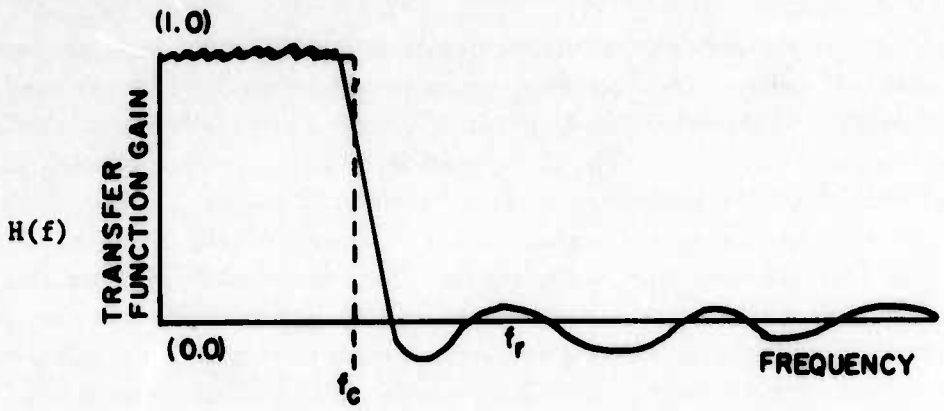
5) Frequency constraining

The straight moving average is comparable to a center point estimate from a first degree polynomial in item (2). The polynomial curve fitting technique gives a single point estimate from a polynomial curve fit to n consecutive data points. The estimate may be end point, center point, or any other point on the curve. The constrained least squares uses the classical approach but adds constraint conditions on the filter parameters based on some known information concerning the data preceding the current filter span. The variable span filter is any of the filter types with the added characteristic that the filter span is adjusted ("opened up" or "closed out" and "reinitialized") based on some characteristics of the input data.

All linear digital filters are frequency-constraining in that they suppress certain frequencies, pass other frequencies without distorting, and may distort or amplify other frequencies.

Because of the low frequency trend characteristics in trajectory data, the filters used to smooth this data are of the "low pass" type. That is, the filter weights are designed to suppress high frequencies and to output the low frequency components with a minimum amount of distortion. In many filters such as the least square polynomial type, the cut-off frequency is determined as a function of the number of points in the filter span. As a number of points in the filter span becomes larger, the cut off frequency becomes lower, i.e., more smoothing is applied to the data. There are other filters, however, which construct the weights for a specific cut-off frequency, and the increase in the filter span does not change the cut-off frequency of the filter, but rather improves the filter's frequency response corresponding to the desired cut-off frequency.

In addition to low-pass filters, weights can be constructed for high-pass, band-pass, and band-reject filters. All filters mentioned thus far can be evaluated by estimating their frequency response function H(f). The frequency response is estimated by taking the Fourier Transform of the weights W_i as a function of i. A typical frequency response versus the ideal frequency response or step function is given on the following page:



An example of some frequency response functions was discussed in Section 4.6 and illustrated in Figure 5. The frequency response function is very important when it comes to error analysis, because knowledge concerning the cut-off frequency, the high-frequency "leakage" (side lobe effect), the sharpness of filter roll-off at the cut-off frequency, and the distortion or amplification of the low frequencies, can indicate whether or not the appropriate filter was used for a given set of data. The frequency response for a finite number of weights will never follow the ideal response curve. The frequency response oscillates or has side lobes at the higher frequencies because the filter is truncated in the time domain, and significant weights W_i have been discarded, causing discontinuities in the time response of the filter. These time discontinuities in the time domain result in the oscillations in the frequency domain at the higher frequencies. To avoid this result, a function must be chosen which decays very quickly in the time domain, so that truncation at a reasonable span length will discard data multipliers of much smaller magnitudes. Slowness of decay in the time domain is caused by discontinuities in the derivatives, particularly the zeroth and other low order derivatives, in the frequency domain. This suggests that, where possible, functions having all continuous derivatives be chosen for use.

The error introduced into the filtering process by the oscillations in the transfer function (curve) is inversely proportional to the number of points used in the filter and the roll-off frequency specified. However, there are means whereby this response can be controlled and a fixed cut off specified, and at the same time have a rapid roll off and very low passage in high frequency areas. Such filters are described in Reference 3.

A discussion of differentiation and prediction filters (see Reference 4) would be too lengthy to pursue in detail. Nevertheless, a few guiding comments will be added here. Two frequency filters with different cut-off frequencies can be "cascaded" in the data reduction process to achieve the digital equivalent of either a band-pass or a band-elimination filter, depending upon the arrangement of the data multipliers and of the filter outputs and residuals. To preserve the high rate of change in slope

and avoid filter lags at the acceleration discontinuities at staging, it becomes necessary to have a variable span filter, and collapse the filter weighting function into the discontinuities and build up the weighting function again on the other side. Unfortunately, such a process does not, in general, preserve the desired qualities of the function. Therefore, if an integration step accompanies the smoothing process, such as occurs in deriving velocity from acceleration, it is necessary either to filter through the discontinuity, thereby sacrificing the high-frequency response of the data, or to perform the integration process first and then filter, collapsing the filter when desired. However, this does require additional computations since the raw data is generally at a higher sample rate than the smoothed data, and in common with the prefiltering approach, careful editing of the raw data is required to prevent integration errors due to large noise spikes in the raw data (see Reference 2).

Another type of frequency digital filtering has become feasible and desirable due to a recently derived algorithm for spectral analysis called the Fast Fourier Transform (FFT) (Reference 27). Besides making digital spectrum analysis more attractive from an economic standpoint, the FFT has enabled many to change their concepts of digital filtering, in that the intellectually appealing approach of filtering in the frequency domain is now often simpler, faster, and just as effective as filtering in the time domain, although two transforms between the time and frequency domain are employed in the process. Suppose, for example, we desire to filter the data $X_1, X_2, \ldots, X_n$ with a low-pass filter at some cut-off frequency f_c. Instead of computing the weights for a time domain convolution we use the FFT to estimate the real and imaginary parts of the spectrum of the time series. Once we obtain the spectrum we truncate it at the appropriate cut-off frequency then take the inverse FFT. The resulting data in the time domain corresponds to the truncated spectrum and, as a result, has been "low pass" filtered with cut-off f_c.

In addition to the so-called general-purpose filters, there are special-purpose digital filters which obtain each filtered data point from the preceding data point using the recursive relationships and information concerning the variance of preceding data contained in the "state vector." The state vector describes the mathematical model of the total system generating the data which is to be filtered. For this filter to function in an optimum manner, the following must be given: (1) system parameters and their configuration, (2) state vector initial values and variance-covariance matrix of state variables, (3) time and effects of special events. The special-purpose filter is much more sophisticated from a mathematical standpoint than the general-purpose or simple digital filter. The special-purpose filter can be a very effective analysis tool especially when used in conjunction with multi-instrument solutions. One of the more well known special-purpose filters is the Kalman filter (see Reference 28).

9.1 EFFECTS OF SMOOTHING ON DATA

There is a diversity of opinion among people who process and analyze data concerning the many numerical methods used in smoothing measured quantities and in the retrieval of information from erroneous observations. The literature on this subject is voluminous. Of the many methods, all use assumptions on the functional form of the basic data trend or the statistical properties or origin of errors in an effort to obtain a numerical process which will, within the imposed constraints, improve the data by minimizing errors. The choice of a technique must depend upon the objectives sought. Some investigation of a given technique's potential in describing the data must be made. For example, statistical smoothing methods depend upon the character of errors, their distribution, their variances, and their dependence or independence. One might choose a second-degree moving arc polynomial over successive 51 point spans of 10 samples per second on position data to achieve a certain reduction in error variance.

Least squares polynomial smoothing has long been used as a general-purpose filter. How well the objectives may be achieved in choosing a smoothing technique may be determined by the knowledge of the functional equations of motion and the error characteristics. Figures 10 thru 17 contain important facts relative to polynomial smoothing and Figure 18 corresponds to a particular set of frequency-constraining digital filters. A description of each follows:

Figure 10 shows estimates of position and velocity error using a polynomial of degree two and various (N_s) smoothing intervals. Evaluation is at midpoint of smoothing interval.

Figure 11 gives the relative smoothing in velocity determinations obtained by using a span of 2n+1 points and evaluating at points other than the midpoint for the case where X can be expressed as a quadratic function of time and the errors in X are uncorrelated.

Figure 12 give curves for higher degree polynomial fits for both velocity and accleration errors.

Figure 17 gives the values

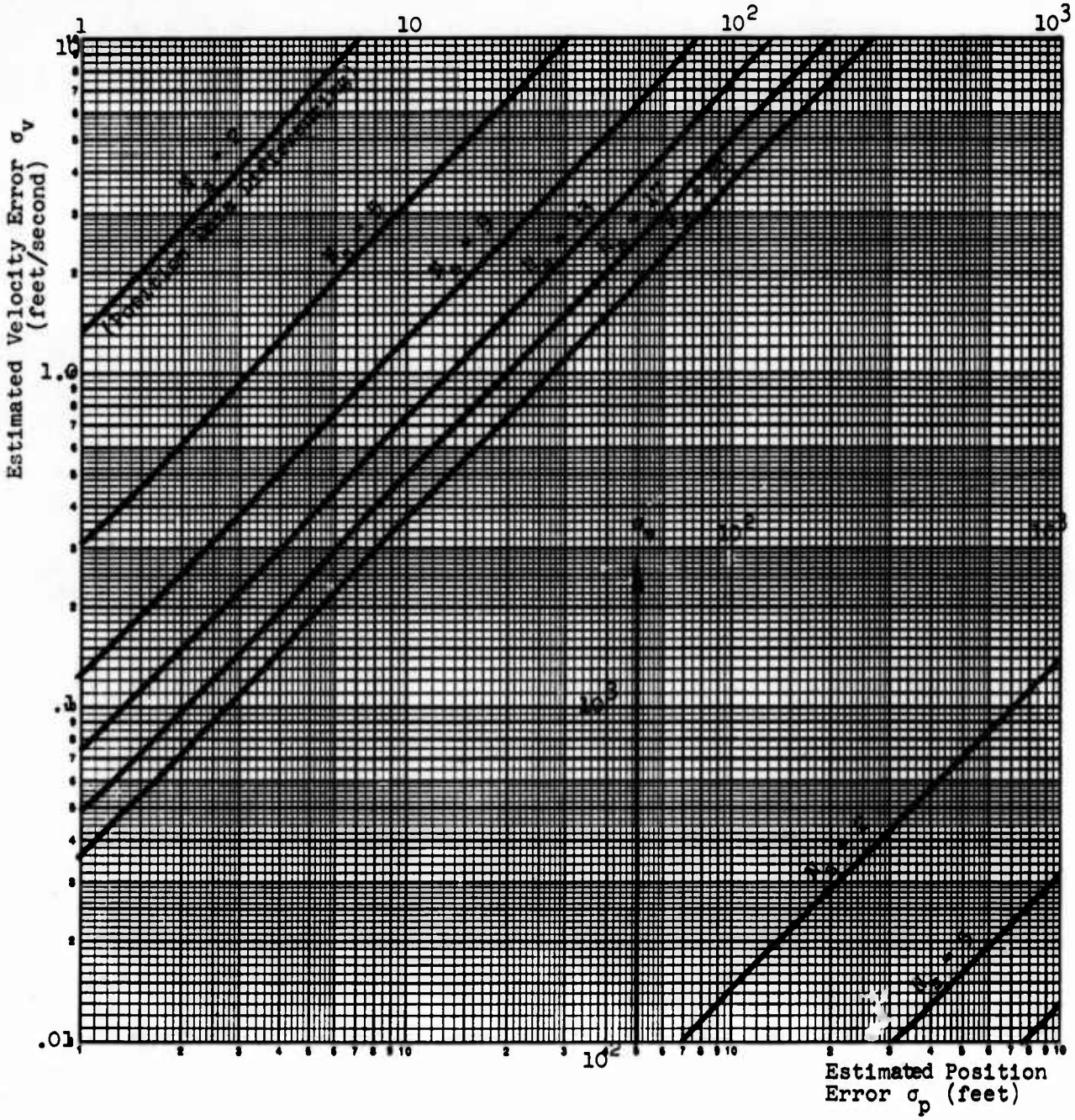
$$K_a = \left(\sum_{j=1}^{m} a_j^2 \right)^{\frac{1}{2}}, \qquad K_b = \left(\sum_{j=1}^{m} b_j^2 \right)^{\frac{1}{2}}, \qquad K_c = \left(\sum_{j=1}^{m} c_j^2 \right)^{\frac{1}{2}}$$

for center point smoothing as a function of the number N of points in the smoothing span.

Figure 18 gives the reduction factor in the standard deviation of the random error of position data. The family of curves represents low-pass filters with spans from 3 to 400 points with cut-off frequencies from 0.0200 hertz to 0.500 hertz.

Estimates of Errors in Velocity Data (σ_v) vs. Errors in Position Data (σ_p) at Midpoints of Dif. Quadratic Smoothing Intervals(N_s)

Figure 10



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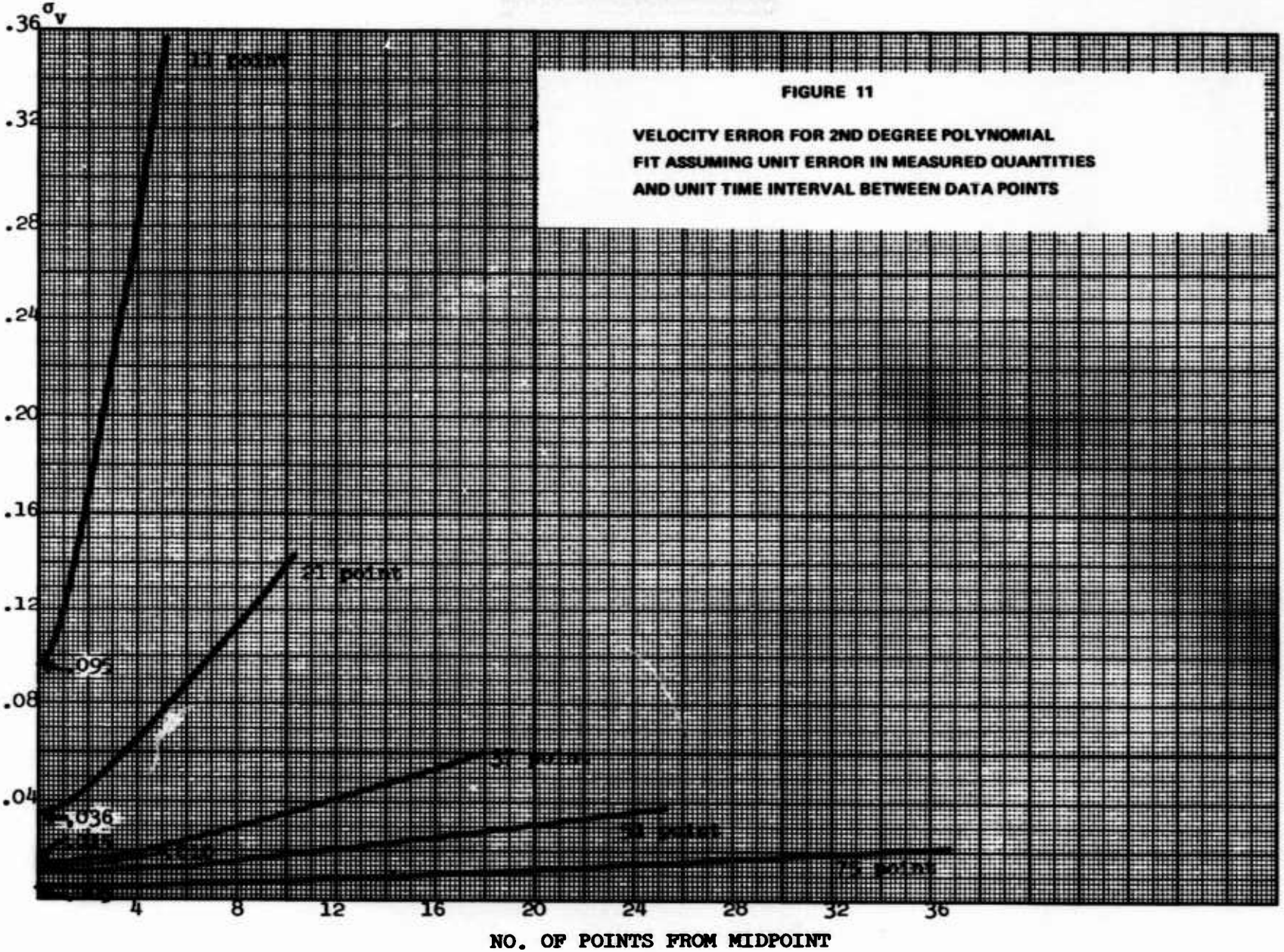


FIGURE 11

VELOCITY ERROR FOR 2ND DEGREE POLYNOMIAL FIT ASSUMING UNIT ERROR IN MEASURED QUANTITIES AND UNIT TIME INTERVAL BETWEEN DATA POINTS

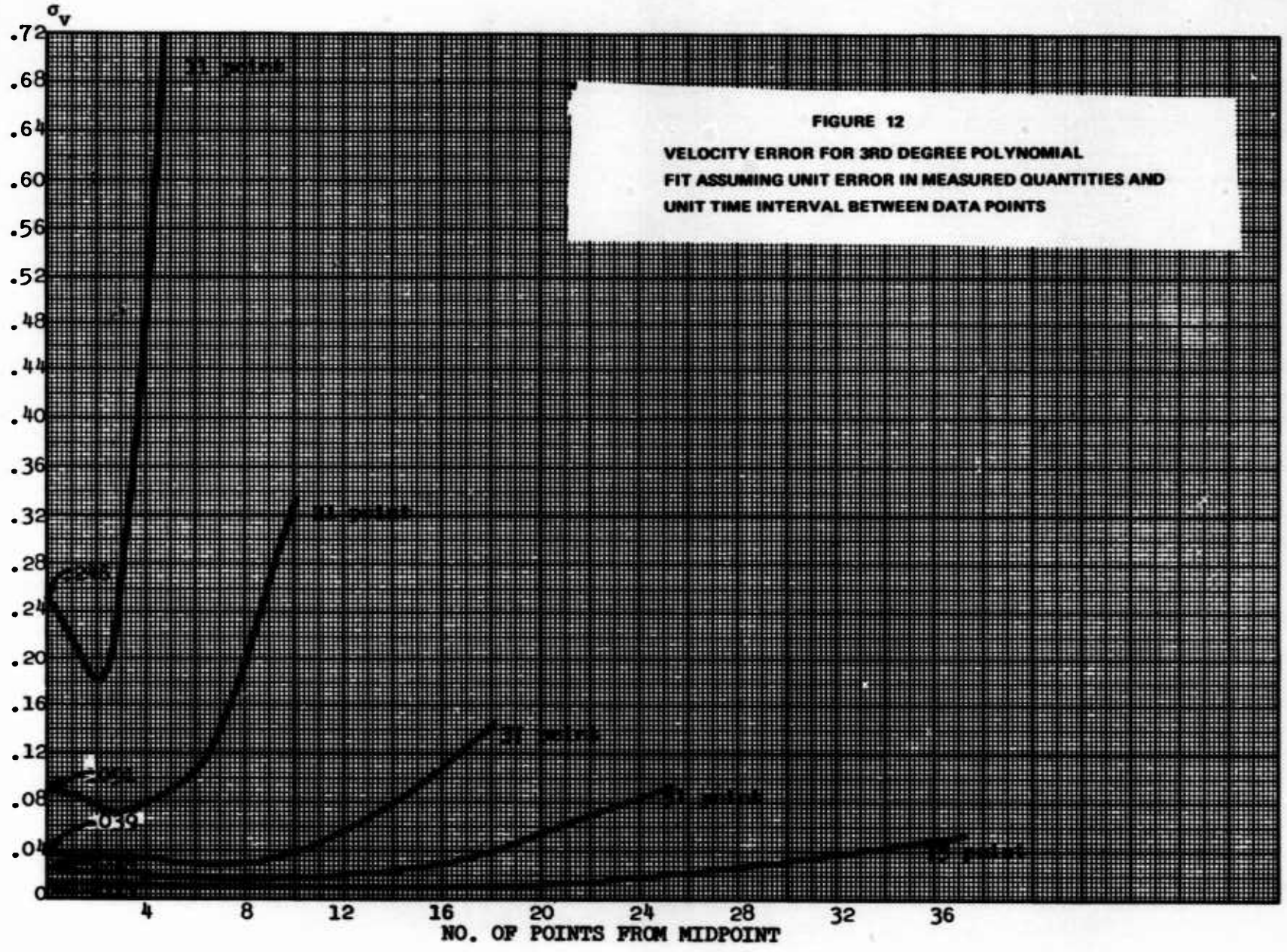


FIGURE 12

VELOCITY ERROR FOR 3RD DEGREE POLYNOMIAL FIT ASSUMING UNIT ERROR IN MEASURED QUANTITIES AND UNIT TIME INTERVAL BETWEEN DATA POINTS

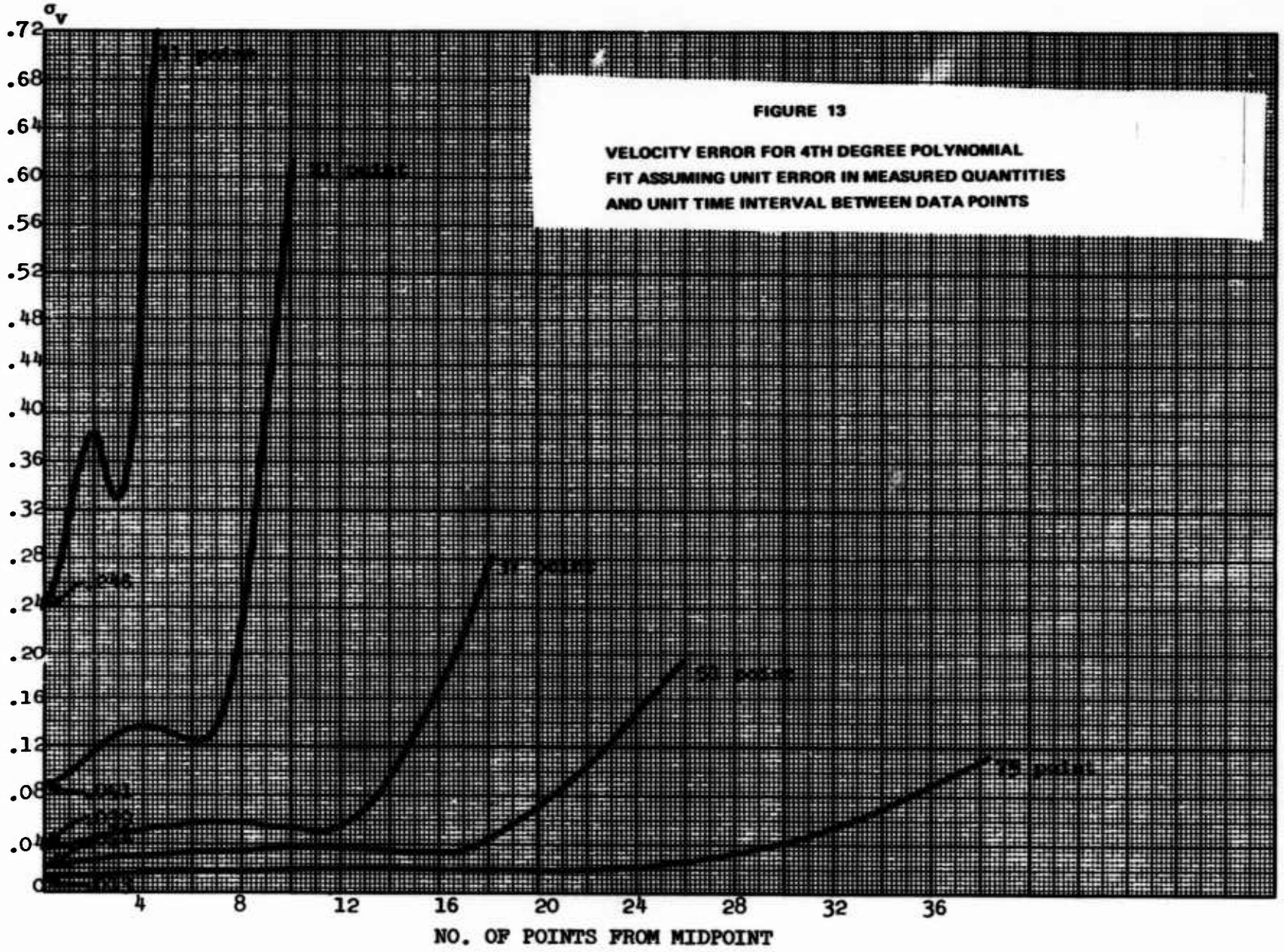


FIGURE 13

VELOCITY ERROR FOR 4TH DEGREE POLYNOMIAL FIT ASSUMING UNIT ERROR IN MEASURED QUANTITIES AND UNIT TIME INTERVAL BETWEEN DATA POINTS

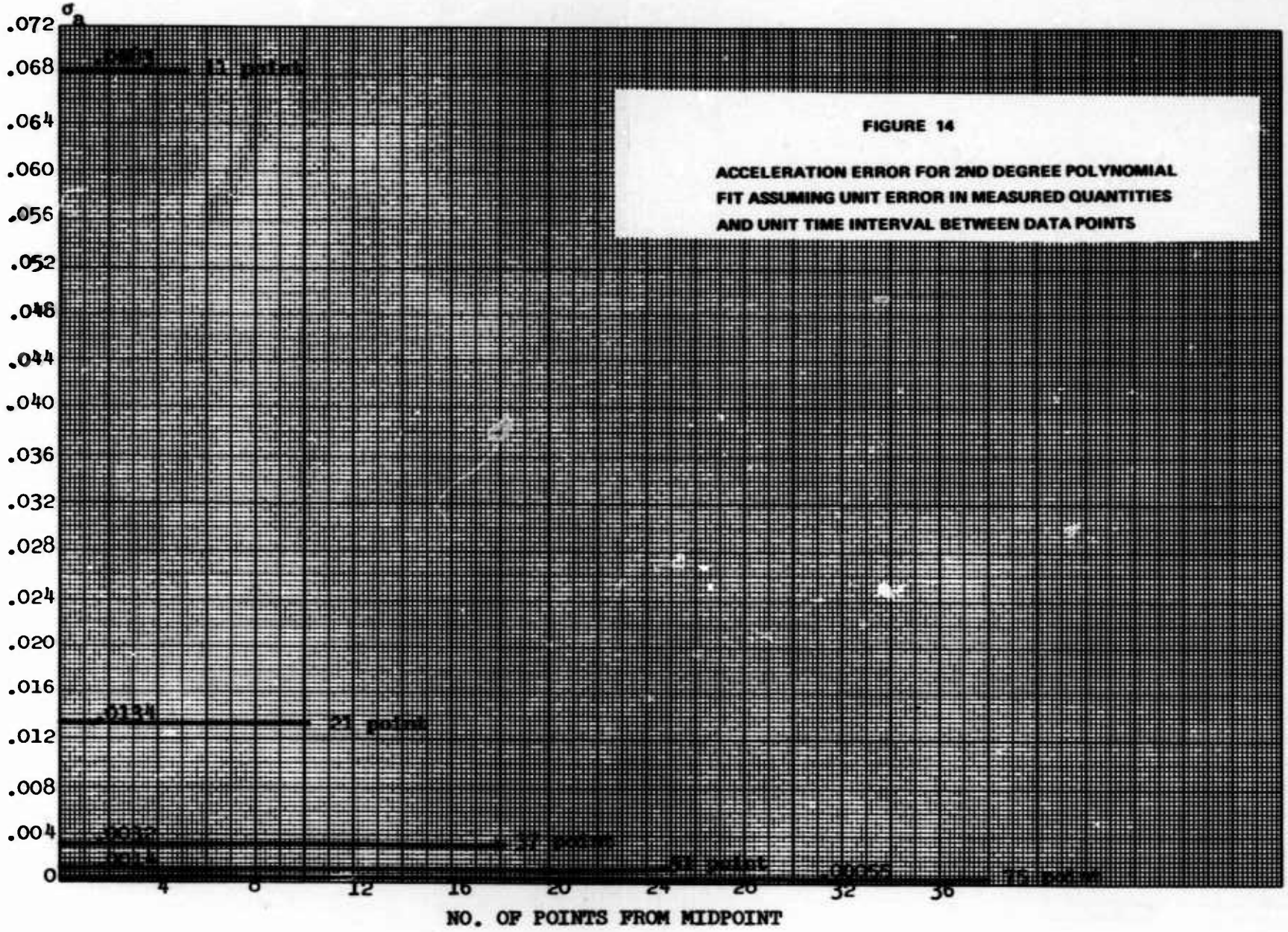


FIGURE 14

ACCELERATION ERROR FOR 2ND DEGREE POLYNOMIAL FIT ASSUMING UNIT ERROR IN MEASURED QUANTITIES AND UNIT TIME INTERVAL BETWEEN DATA POINTS

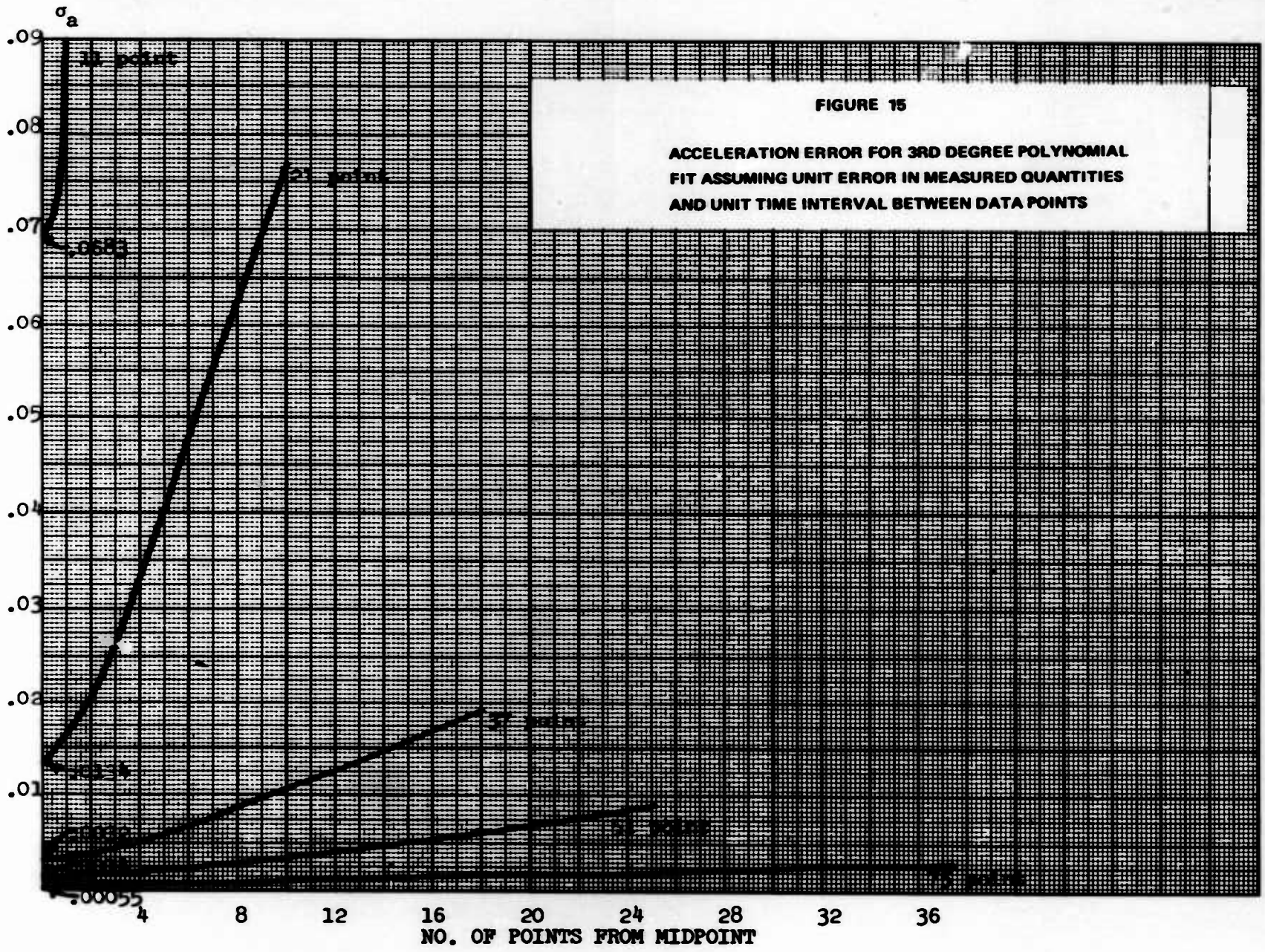


FIGURE 15

ACCELERATION ERROR FOR 3RD DEGREE POLYNOMIAL FIT ASSUMING UNIT ERROR IN MEASURED QUANTITIES AND UNIT TIME INTERVAL BETWEEN DATA POINTS

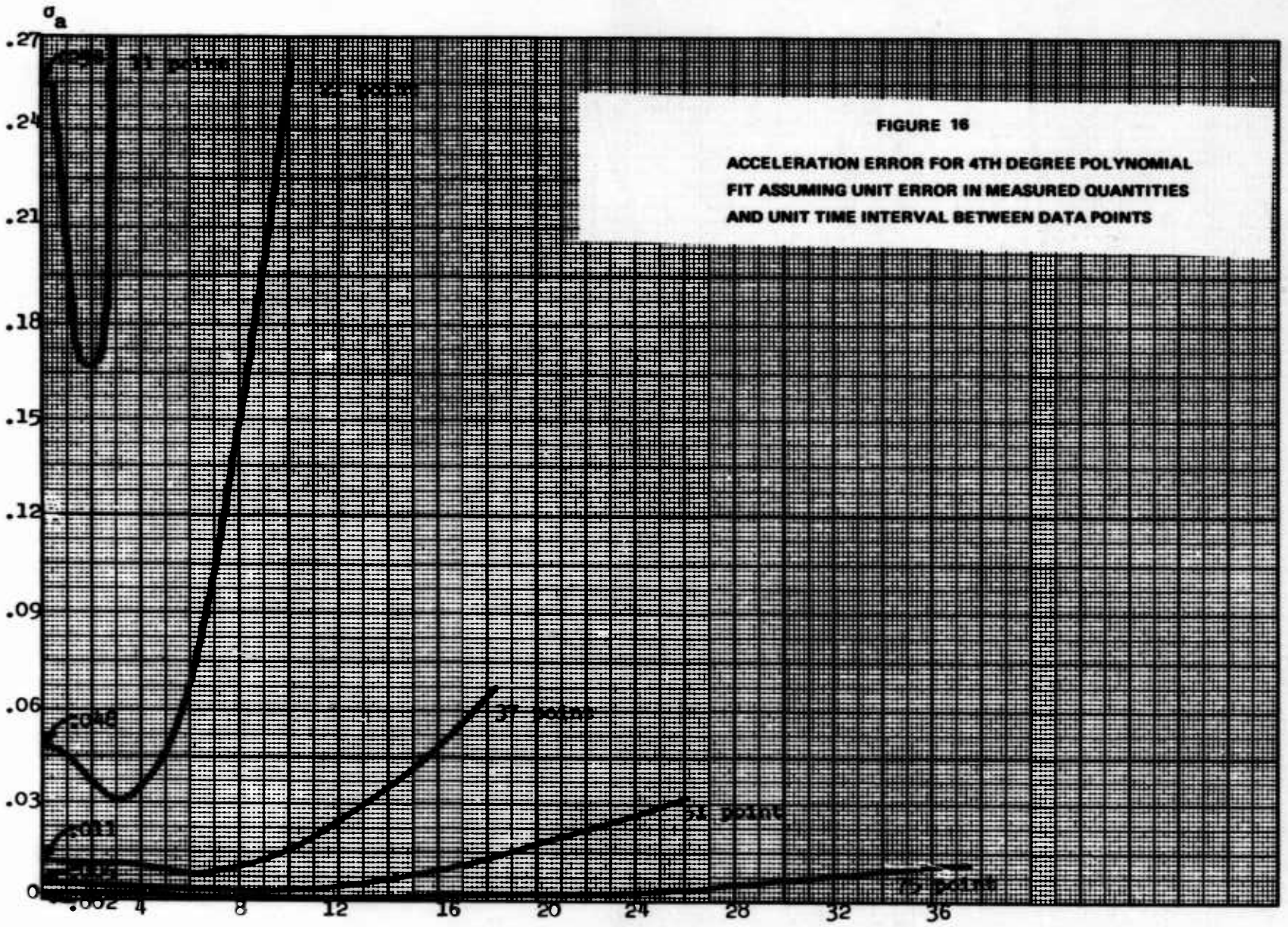


FIGURE 16

ACCELERATION ERROR FOR 4TH DEGREE POLYNOMIAL FIT ASSUMING UNIT ERROR IN MEASURED QUANTITIES AND UNIT TIME INTERVAL BETWEEN DATA POINTS

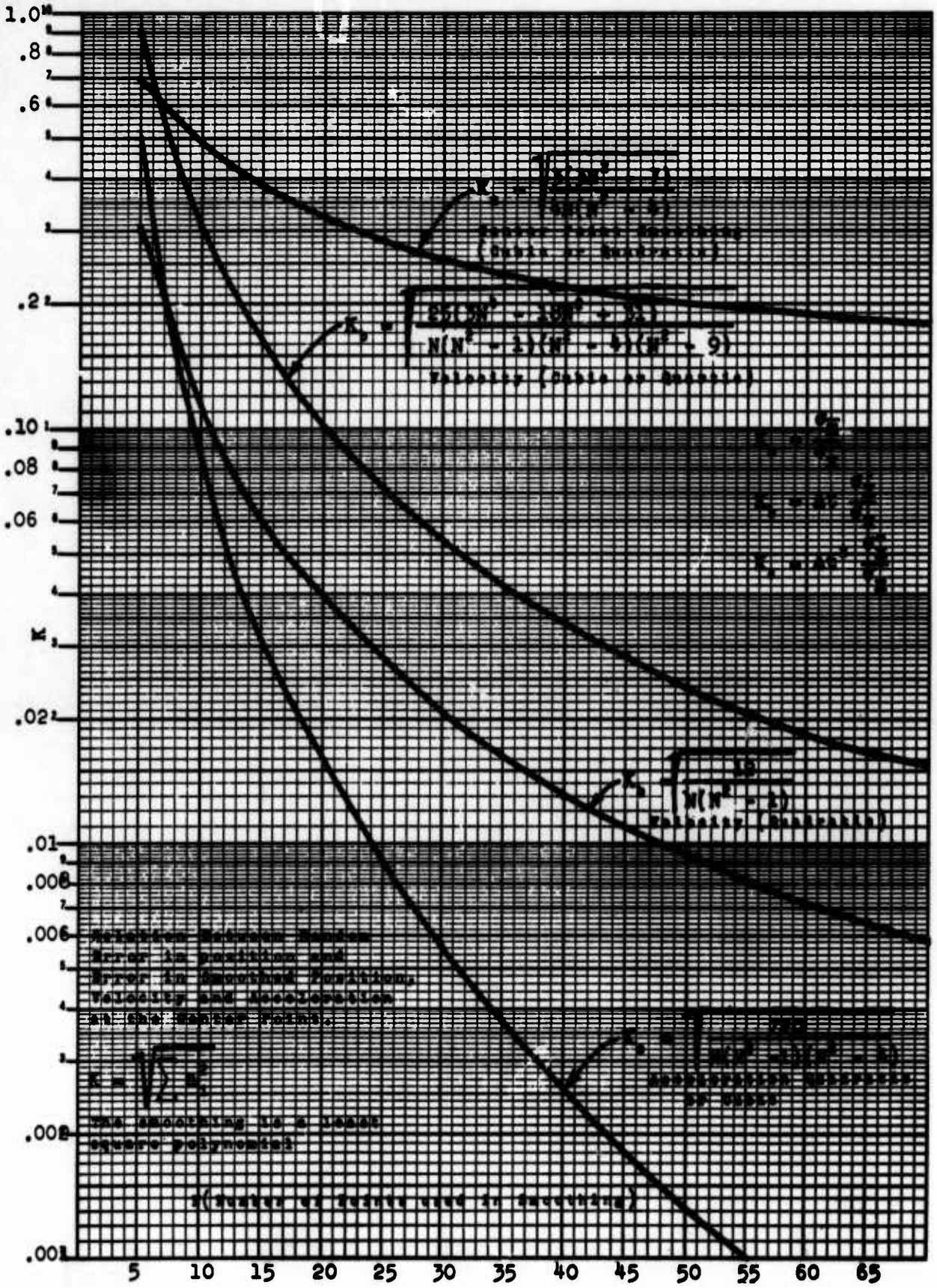


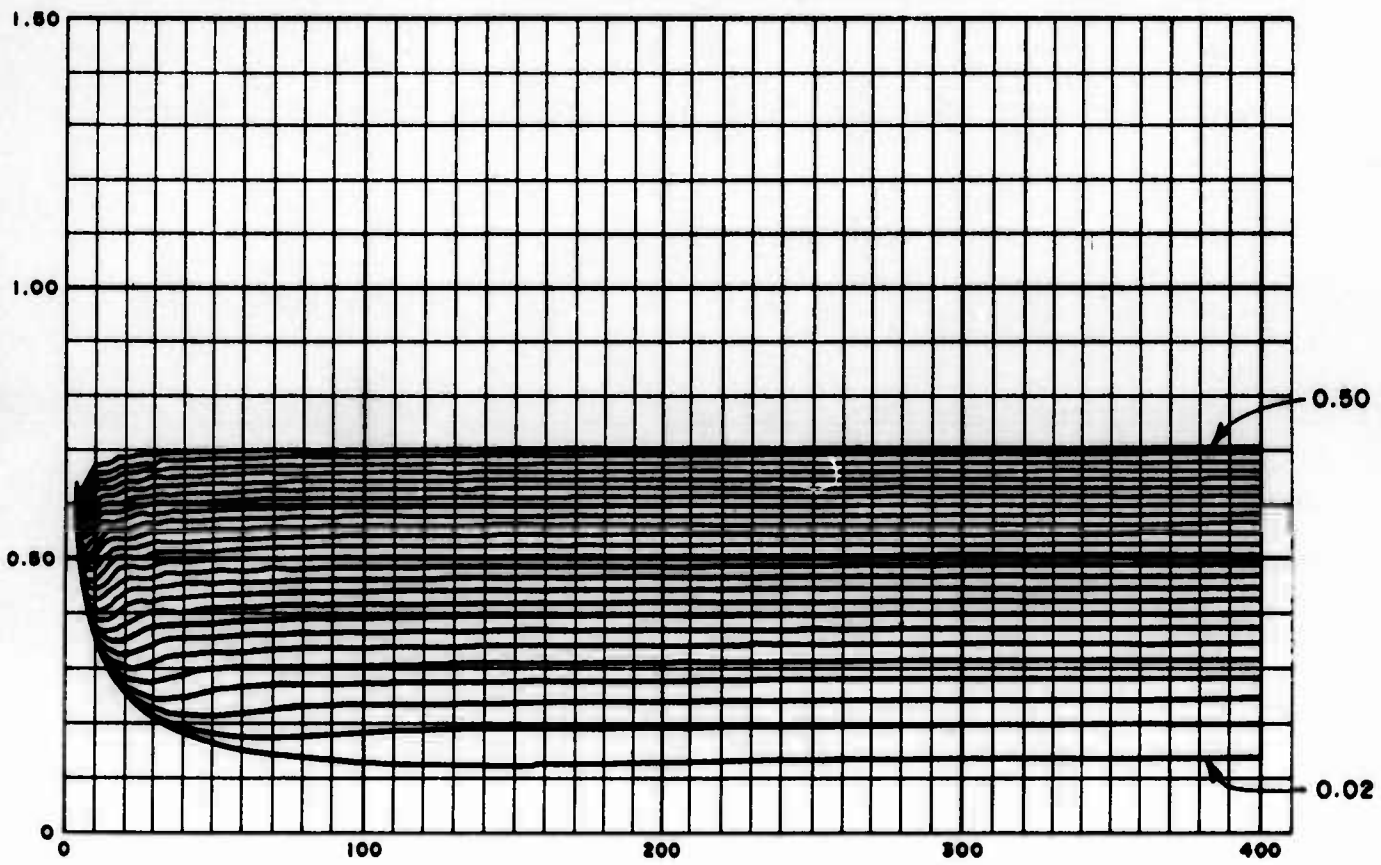
Figure 17 K_a, K_b, and K_c as a function of N

FIGURE 18

VARIANCE REDUCTION CURVE FOR WEIGHTING FACTOR & NUMBER

FILTER TYPE = LOW

CUT-OFF FREQ = .0200 .1400 .2600 .3800 .5000
.0400 .1600 .2800 .4000
.0600 .1800 .3000 .4200
.0800 .2000 .3200 .4400
.1000 .2200 .3400 .4600
.1200 .2400 .3600 .4800



10.0 INERTIAL GUIDANCE DATA

Traditionally, attention and effort have been centered upon attaining velocity and accleration data from position data. Techniques for determining the accuracies of the derived quantities have been discussed. Recently, there has been a development of strong interest in utilizing data from accelerometers of guidance systems to assess velocity accuracies. Historically, this may appear to be a paradox. Development of the many tracking systems on the various ranges had, as a primary objective, the evaluation of guidance systems. Hence, it seems odd now to consider the use of guidance systems to evaluate tracking systems. However, the weight of accumulated evidence and experience has forced the conclusion that the quality, performance, and reliability of these systems are high. Nevertheless, the simple integration of acceleration data to produce velocities has an inherent disadvantage in that additional unknowns are introduced.

In view of the above, techniques have been developed for combining this data with that of other sources to establish trajectories. It has been suggested that errors in velocity arise from accelerometer errors such as bias, scale factor, quadratic nonlinearity, platform misalignment, gyro mass unbalance, and gyro constant drift. Errors from these sources are relatively small during the initial portions of flight, but after 50 seconds or so tend to exceed those present in data from the better CW Doppler interferometers. For a typical system, these errors have been identified as follows, in the order listed:

$$
\begin{aligned}
\Delta\dot{X}_0 &= K_0 t \quad , \quad K_0 = 30 \times 10^{-6} g \\
\Delta\dot{X}_1 &= K_1 \dot{X} \quad , \quad K_1 = 10 \text{ ppm} \\
\Delta\dot{X}_2 &= K_2 / a_x^2 dt \quad , \quad K_2 = 3 \text{ ppm/g} \\
\Delta X_3 &= K_3 \dot{Z} \quad , \quad K_3 = 2 \text{ sec.} \\
\Delta\dot{X}_4 &= K_4 \dot{Z}^2 \quad , \quad K_4 = 0.01 \text{ deg/hr/g} \\
\Delta\dot{X}_5 &= K_5 a_z t dt \quad , \quad K_5 = 0.05 \text{ deg/hr.}
\end{aligned}
\qquad (10.0.1)
$$

It may be observed that few of these are identifiable in functional form and in certain cases a high degree of correlation is present. Furthermore, some of these are time dependent and tend to grow with time. A factor not appearing above but one exerting considerable influence on the analysis of velocity is the propagation of tracking noise into the velocity vector. This also tends to grow with time, and hence, tends to obliterate the data towards the end of powered flight. It should be noted that on conventional solid-propellant vehicles, loss of guidance data is usual following missile burnout because of the dynamics of the vehicle attendant to thrust termination. Consequently, no free-fall flight data are ordinarily available.

For powered flight data analysis, velocity error estimates may be obtained directly by comparing the velocity data derived from the tracking systems positional data with inertial guidance velocity data. Inertial guidance data are subject to long term drift errors, but exhibit good characteristics in the

very low frequency errors (errors below, .05 cycles per second); whereas, the data from external systems are subject to errors in the high frequency and middle frequency regions (errors above .05 cycles per second), but exhibit good characteristics in the low frequency or near zero frequency region. It is this very fact, i.e., that these differences exist in the frequency characteristics of the two systems, that the guidance data may be used as a standard for noise estimation in the external data, and the external data, in turn, may be used as a standard for the estimation of long term drifts or biases in the guidance data.

The analysis of free-fall data may proceed along two lines. It may be assumed that the observed X, Y, Z data are serially correlated and processed through smoothing filters in order to obtain velocity data ($\dot{X}$, $\dot{Y}$, $\dot{Z}$) and acceleration data ($\ddot{X}$, $\ddot{Y}$, $\ddot{Z}$). Then estimates of the error in the derivative data may be obtained as a function of the variance estimate of the random + cyclic error in position. The position error variance which propagates into velocity and acceleration is a composite of the high and low frequency noise in the data. This error is also referred to as the random + correlated error, or perhaps most appropriately described as the total error minus the constant bias error. Constant bias errors in the position data introduce negligible effects in the derivative velocity and acceleration data, since by their very nature they tend to disappear in the process of differentiation. It is (cyclic) low frequency error which has the most deleterious effect on the quality of the derivative data. Smoothing over spans appreciably shorter than the typical low frequency does little to reduce its influence. The rigorous propagation of correlated errors through any specified smoothing filter requires a knowledge of the autocorrelation function of the errors. If the autocorrelation coefficients for successive values of X and 1, P_1, P_2, P_3..., the variance of the value of X obtained from the smoothing filter

$$X_i = \sum_{j=-m}^{m} b_j X_{i+j} \qquad (10.0.2)$$

is given by

$$\sigma^2_{x_i} = \sigma^2_x \, b \, R \, b^T \qquad (10.0.3)$$

in which b is the row vector of data multipliers

$$b = (b_{-m} \; b_{-m+1} \; \cdots \; b_{-1} b_0 \; b_1 \cdots \; b_{m-1} \; b_m) \qquad (10.0.4)$$

R is the auto-correlation matrix:

$$R = \begin{bmatrix} 1 & P_1 & P_2 & P_3 & \cdots & P_{2m} \\ P_1 & 1 & P_1 & P_2 & \cdots & P_{2m-1} \\ P_2 & P_1 & 1 & P_1 & \cdots & P_{2m-2} \\ P_3 & P_2 & P_1 & 1 & \cdots & P_{2m-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ P_{2m} & P_{2m-1} & P_{2m-2} & P_{2m-3} & & 1 \end{bmatrix} \qquad (10.0.5)$$

and σ_x^2 is the variance of X_i.

Similarly, the variance of $\ddot{X}$ obtained from the smoothing filter

$$\ddot{X}_i = \sum_{j=-m}^{m} c_j X_{i+j} \qquad (10.0.6)$$

is given by

$$\sigma_{\ddot{X}_i}^2 = \sigma_x^2 \, c \, R \, c^T \qquad (10.0.7)$$

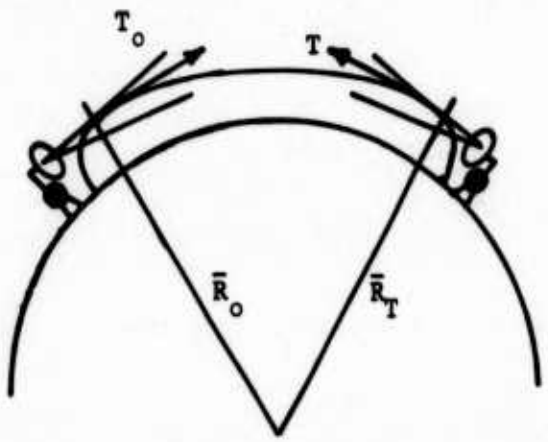
in which c is the row vector of data multipliers

$$c = (c_{-m} c_{-m+1} \cdots c_{m-1} \; c_m), \qquad (10.0.8)$$

R is the autocorrelation matrix.

An estimate of the autocorrelation function is best obtained from the residuals about a free fall ellipse (Keplerian ellipse) fitted to a long span (preferably 100 to 200 seconds) of post burnout data. The autocorrelation function may be obtained from the power spectrum analysis computer program.

There have been many difficulties associated with the use of aircraft and ballistic camera coverage on missile tests to calibrate external tracking system velocity errors to acceptable accuracies, which have led to the use of a second technique. This method consists of fitting the Keplerian equations of motion to data taken at both ends of the free flight trajectory, i.e., data taken by an up-range tracking system of the missile position after missile burnout and data taken by a down-range tracking system of missile position prior to atmospheric re-entry. Position parameters are fitted to a Taylor series expansion of the equations of motion truncated at the fourth power. Because the time between the two data spans is accurate, velocities obtained from the trajectory reconstructed in this manner will also be accurate (typically on the order of 0.1 fps) although there may be sizeable position uncertainties (typically 200-300 feet) in the data taken at each end.



At $t = t_o$

$$\bar{R}_T = \bar{R}_o + \bar{V}_o(T - t_o) + \int_{t_o}^{T} \int_{t_o}^{t} \bar{g}\, dtdt. \qquad (10.0.9)$$

Hence,

$$V_o = \frac{\bar{R}_T - \bar{R}_o}{T - t_o} - \frac{1}{T - t_o} \int_{t_o}^{T} \int_{t_o}^{t} \bar{g}\, dtdt. \qquad (10.0.10)$$

The derivation of this result is not difficult. From the Figure, $\bar{R}_0$ is the radius at injection time t_0 and T is time of re-entry where radius is $\bar{R}_T$. Since this portion of the trajectory is substantially outside the atmosphere the use of no-drag formulation with only gravity force acting is justified. Thus, acceleration is:

$$a = \bar{g}$$

or

$$\frac{dv}{dt} = \bar{g}$$

$$dv = \bar{g}\, dt \qquad (10.0.11)$$

$$V = \int_{t_0}^{T} \bar{g}\, dt + C_1$$

and

$$R = V_0(T - t_0) + \int_{t_0}^{T} \int_{t_0}^{T} \bar{g}\, dtdt + C_2$$

or (10.0.12)

$$\bar{R}_T = R_0 + V_0(T - t_0) + \int_{t_0}^{T} \int_{t_0}^{T} \bar{g}\, dtdt.$$

This may then be solved for V_0.

11.0 ERROR ANALYSIS IN THE FREQUENCY DOMAIN

A very effective tool to use in the evaluation of error is spectral analysis. It is assumed that the errors e_t are a realization from a stochastic process $\{\epsilon_t\}$ which is stationary in the wide sense. That is, the process has a constant mean and its autocovariance is independent of time, but is dependent on time lag. That is

$$E(\epsilon_t) = \mu(t) = \text{CONSTANT}$$

and (11.0.1)

$$R_{\epsilon_t}(k) = E[(\epsilon_t - \mu)(\epsilon_{t+k} - \mu)], \quad k = 0,1,2,\ldots,m.$$

The assumption of stationarity is generally true over subsets of the data, and if ϵ_t is not stationary it can usually be "prewhitened" or "detrended" so that nonstationary effects can be removed from the data. The variance of ϵ_t is obtained from R(k) when k = 0, i.e., $\sigma_{\epsilon_t}^2 = R(O)$. The autocorrelation function of ϵ_t is given by

$$\rho(k) = \frac{R(k)}{R(0)}.$$

It can be shown that if ϵ_t is a stationary process then the autocovariance has the following properties

$$\begin{aligned} &R(0) > 0 \\ &R(-k) = R(k) \\ &|R(k)| \leq R(0) \\ &\lim_{k\to\infty} R(k) = 0 \end{aligned} \qquad (11.0.2)$$

Further, it can be shown that the autocorrelation matrix

$$[\rho] = \begin{bmatrix} \rho(o) & \rho(1) & \rho(2) & \ldots & \rho(p) \\ \rho(-1) & \rho(0) & \rho(1) & \ldots & \rho(p-1) \\ \vdots & \vdots & \vdots & & \vdots \\ \rho(-p) & \rho(-p+1) & & \ldots & \rho(0) \end{bmatrix} \qquad (11.0.3)$$

is positive definite

The spectral density function for continuous ε_t is

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau) d\tau \qquad (11.0.4)$$

and if ε_t is discrete (stochastic sequence) then

$$f(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\tau=\infty} e^{-i\omega\tau} R(\tau). \qquad (11.0.5)$$

Conversely, if we know the spectral density $f(\omega)$ then

$$R(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} f(\omega) d\omega \qquad (11.0.6)$$

and

$$R(\tau) = \int_{-\pi}^{\pi} e^{i\omega\tau} f(\omega) d\omega \qquad (11.0.7)$$

for the continuous and discrete cases respectively. In digital analysis it is clear we are interested in using the discrete case to approximate the continuous function, and from (11.0.7) we can see that the quantities

(11.0.8)

$$\frac{1}{2\pi} R(-\tau), \quad (\tau = 0, \pm 1, \pm 2, \ldots),$$

are the Fourier coefficients of the function $f(\omega)$. In actuality ϵ_t is generally continuous but a realization e_t is usually discrete. In addition to the spectral density function, another important function in the frequency domain is the spectral distribution function

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\omega)d\omega \tag{11.0.9}$$

which distributes the variance or power as a function of frequency increments. If the Fourier Transform of the autocorrelation function is taken, then the total area under the spectral density curve is unity. If the Fourier Transform of the autocovariance is taken, then the total area under the curve is $R(o) = \sigma^2$ (in this case the spectral density is often called the power spectrum).

In actual practice the autocovariance is approximated from n error estimates $e_1,\ldots,\epsilon_n$ over a total of $m < n$ lags.

$$\hat{R}_e^{(p)} = \frac{1}{n-p} \sum_{i=1}^{n-p} (e_i - \bar{e})(e_{i+p} - \bar{e}),\ p = 0,1,2,\ldots,M \tag{11.0.10}$$

where

$$R_e^{(p)} = R_e(p\Delta t),\ \Delta t = \text{time increment.}$$

The "raw" spectral density is estimated by numerically estimating the Fourier transform of $R_e^{(p)}$ using the trapezoidal rule

$$\hat{f}(s) = \frac{2\Delta t}{\pi} \sum_{p=0}^{M} \varepsilon_p R_e^{(p)} \cos \frac{sp\pi}{M},\ s=0,1,\ldots,M$$

where

$$\varepsilon_p = \begin{cases} 1, & 0 < p < M \\ 1/2, & p = 0, M \end{cases} \tag{11.0.11}$$

$$\hat{f}(s) = \hat{f}(\omega_s) = \hat{f}\left(\frac{s\pi}{M\Delta t}\right)$$

The raw spectral estimates are smoothed using a Hamming spectral window. In the frequency domain the Hamming weights give a three-point moving average and the filtered spectrum estimates are

$$\bar{f}(0) = .54\ \hat{f}(0) + .46\ \hat{f}(1)$$

$$\bar{f}(s) = .23\ \hat{f}(s-1) + .54\ \hat{f}(s) + .23\ \hat{f}(s+1) \qquad (11.0.12)$$

$$\bar{f}(M) = .54\ \hat{f}(M) + .46\ \hat{f}(M-1)$$

The Hamming weights are used to reduce the side band effect caused by discrete approximation when frequencies are not exactly on the discrete spectral estimate point. The frequency response of the Hamming weights and others are well known and there is much literature on the subject. Since

$$\omega_s = \frac{s\pi}{M\Delta t} = 2\pi f_s,$$

then the spectral density estimates will be in increments of $\Delta f = \frac{1}{2M\Delta t}$ for $0 \leq f \leq \frac{1}{2\Delta t}$. The frequency $\frac{1}{2\Delta t}$ is called the Nyquist or "folding frequency," f_N, and is the maximum frequency which can be resolved for a given sampling rate Δt. The Nyquist frequency is called the folding frequency because it can be shown that if significant frequencies exist in the data which are higher than $\frac{1}{2\Delta t}$ hertz, they will "fold" about the Nyquist frequency and appear on the spectrum as a lower frequency between 0 and $\frac{1}{2\Delta t}$ hertz. This is illustrated in Figure 19 below. For example, suppose ϵ_t is a continuous (analog) function which contains a 60 cycle "hum" component from a power source and it is digitized at $\Delta t = .05$ seconds. $\frac{1}{2\Delta t}$ = 10 cycles = f_N and the 60 cycle component would falsely appear as a zero frequency component or bias in ϵ_t. The folding of higher frequencies is called aliasing.

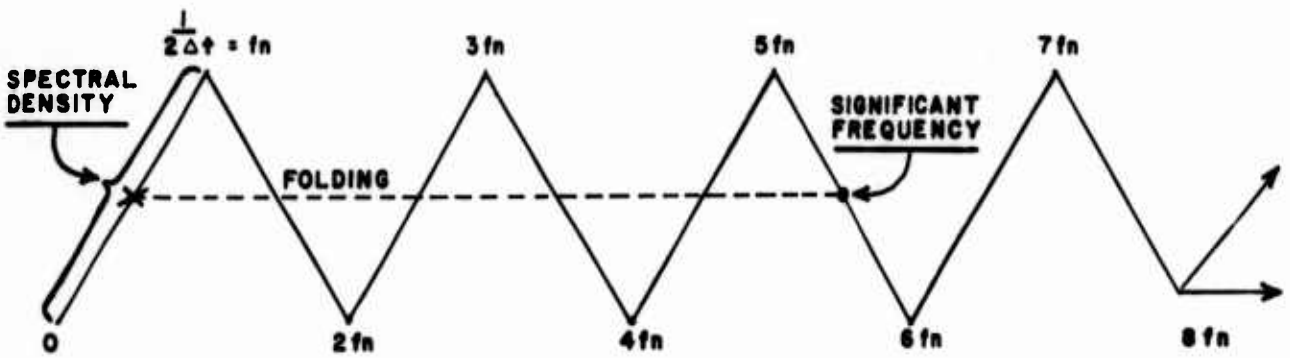


FIGURE 19 EFFECTS OF FOLDING OR ALIASING

The concept of aliasing immediately points to the fact that if the data are sampled with Δt such that $\frac{1}{2\Delta t}$ < significant frequencies, erroneous frequencies will appear in the data. The effects of these frequencies on the error in position, velocity and acceleration data can be highly significant. If the data are such that a large Δt (low sampling rate) is required, then one must be assured that significant frequencies do not exist above the Nyquist frequency. This can be assured by prefiltering the data using a higher sampling rate and a low pass filter with a cut-off frequency equal to the Nyquist frequency corresponding to the desired low sampling rate. This is sometimes necessary when spectral analysis on low frequency error components is to be accomplished.

Since the relation between the autocovariance and the spectral density function is known, we can utilize these two functions in the analysis of many types of data and for error analysis specifically. For this, several examples of autocovariance and spectral density functions corresponding to different error models are given (see Figure 20).

Suppose $e_1,\ldots,e_n$ is a realization from a random process which is independent (uncorrelated). Since the e_i are discrete, the realization will have a low pass frequency band and since the realization is uncorrelated, it is called white noise. In this case

$$R(k) = \begin{cases} \sigma^2, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{11.0.13}$$

Then, according to (11.0.2) the corresponding spectral density is

$f(\omega)=\frac{\sigma^2}{2\pi}$. Thus stationary sequences of uncorrelated random errors are characterized by the fact that their spectral densities are constant over the interval $\omega = 0$ to $\omega = \pi/\Delta t$. An example of the estimated correlation of white noise from radar data is given in Figure 21.

Suppose now that the realization $e_1, e_2, \ldots, e_n$ is correlated with autocovariance

$$R(k) = \sigma^2 \rho^k, \quad |e| < 1. \tag{11.0.14}$$

Autocorrelation is $\frac{R(k)}{R(0)} = \rho^k$ and this form of correlation corresponds to a first order Markov process, i.e., $\epsilon_t = \rho\, \epsilon_{t-1} + \eta_t$. The autocorrelation decays exponentially and from (11.0.5) it can be shown that

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{1 - e^2}{|e^{i\omega} - e|^2} \tag{11.0.15}$$

which is an exponential type spectral density function (see Figure 20).

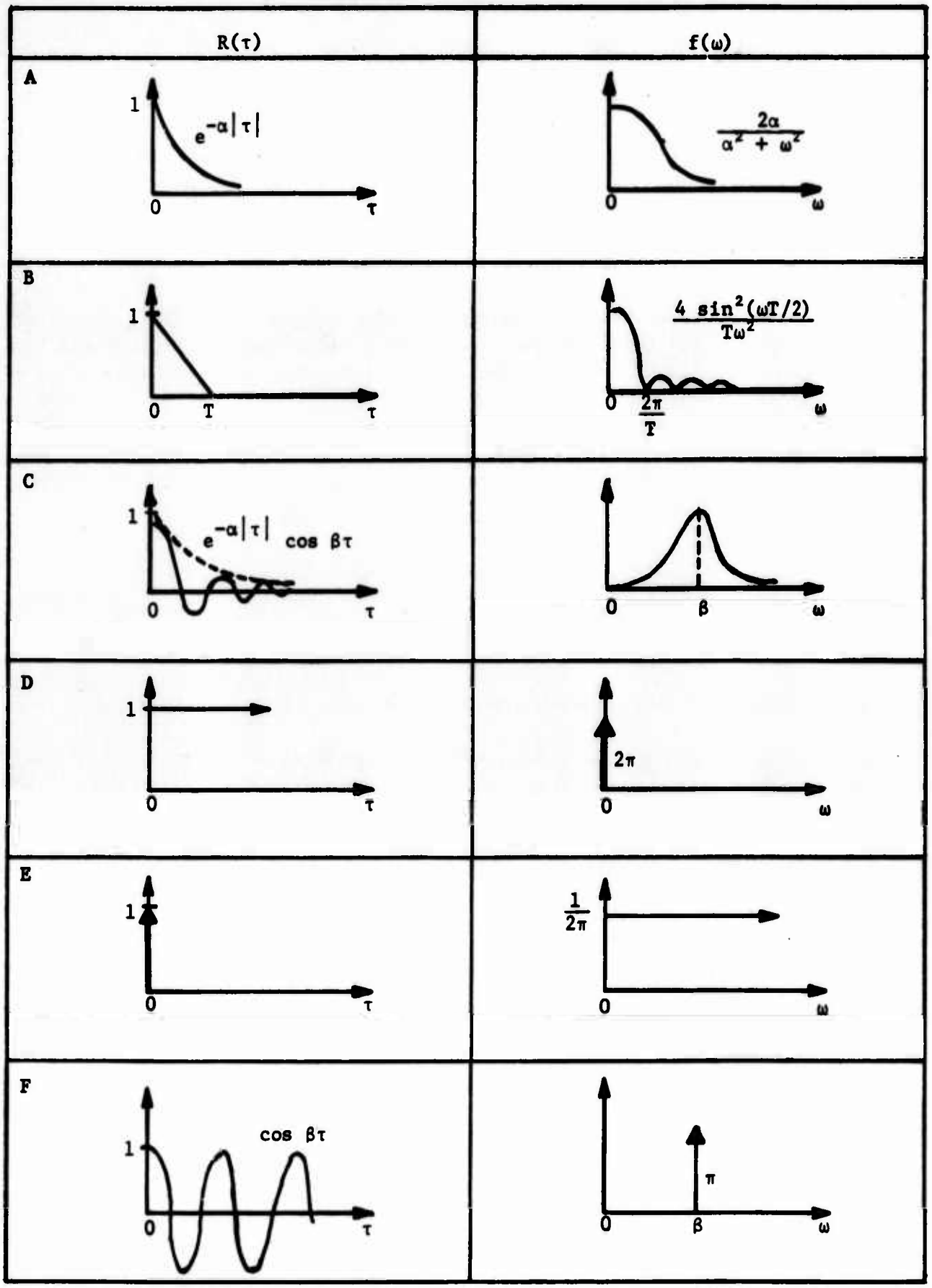


Figure 20 Examples of some continuous autocorrelation functions and corresponding spectral density functions. Types A and C are the continuous analogs to a first and second order Markov Process respectively.

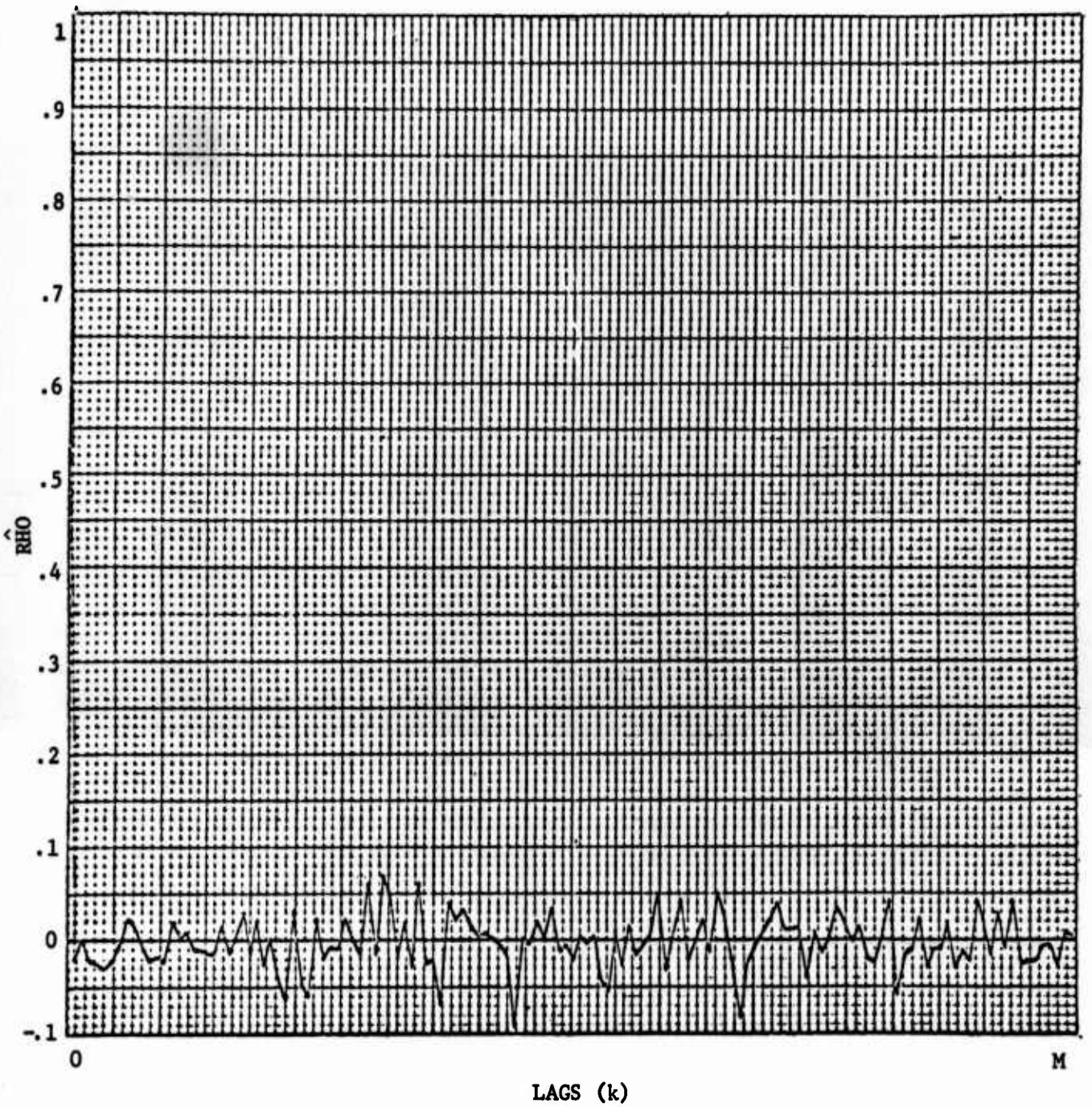


FIGURE 21 Estimate of autocorrelation function of independent random error (white noise).

Markov processes can be estimated by the so-called autoregressive scheme. For example, the autocorrelation and the power spectrum can be estimated for a particular set of error estimates. If the shape of the autocorrelation and power spectrum indicate that the underlying stochastic process is Markov, it can be approximated by an autoregressive scheme. To accomplish this, the error estimates are staggered in the following manner

e_5	e_4	e_3	e_2	e_1
e_6	e_5	e_4	e_3	e_2
e_7	e_6	e_5	e_4	e_3
⋮	⋮	⋮	⋮	⋮
e_n	e_{n-1}	e_{n-2}	e_{n-3}	e_{n-4}

The first column is considered to be the dependent stochastic variable. The relation $e_i = \alpha_1 e_{i-1} + \alpha_2 e_{i-2} + \alpha_3 e_{i-3} + \alpha_4 e_{i-4} + \eta_i$ is then estimated by regression techniques. An analysis of variance can indicate which coefficients are significant, and then the process can be repeated using the appropriate order for the Markov process. An example of such a technique is illustrated in the autoregressive fit in Figure 23. The corresponding autocorrelation estimate is in Figure 22. An analysis of variance corresponding to the autoregressive scheme is tabled in Figure 24. The analysis of variance indicates the process is of the first order because the regression coefficients a_2, a_3, a_4, are not significant. Figure 23 gives the graph of the approximation of η_i. If $e_1,\ldots,e_n$ is a realization from the process $\epsilon_t = a_1\epsilon_{t-1} + a_2\epsilon_{t+2} + \eta_t$, η_t is independent random variable, then it can be shown the autocorrelation function is an exponentially damped harmonic of the form.

$$\rho_k = \left[a_2^{k/2} \sin (kV + W)\right] / \sin W \qquad (11.0.16)$$

where

$$V = \cos^{-1}\left[\frac{-a_1}{2a_2^{1/2}}\right]$$

$$W = \tan^{-1}\left[\frac{(1 + a_2)\tan V}{1 - a_2}\right]$$

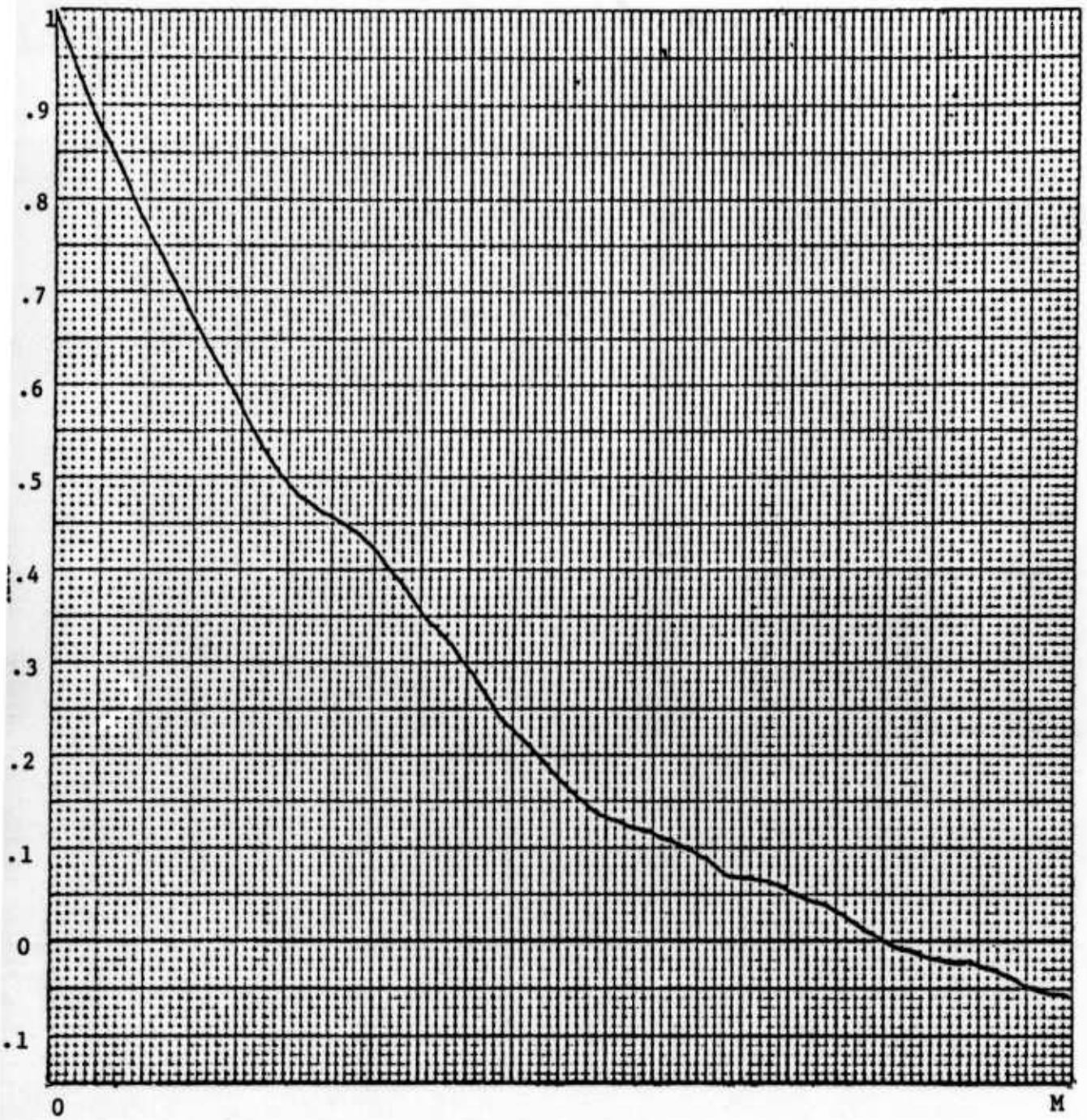
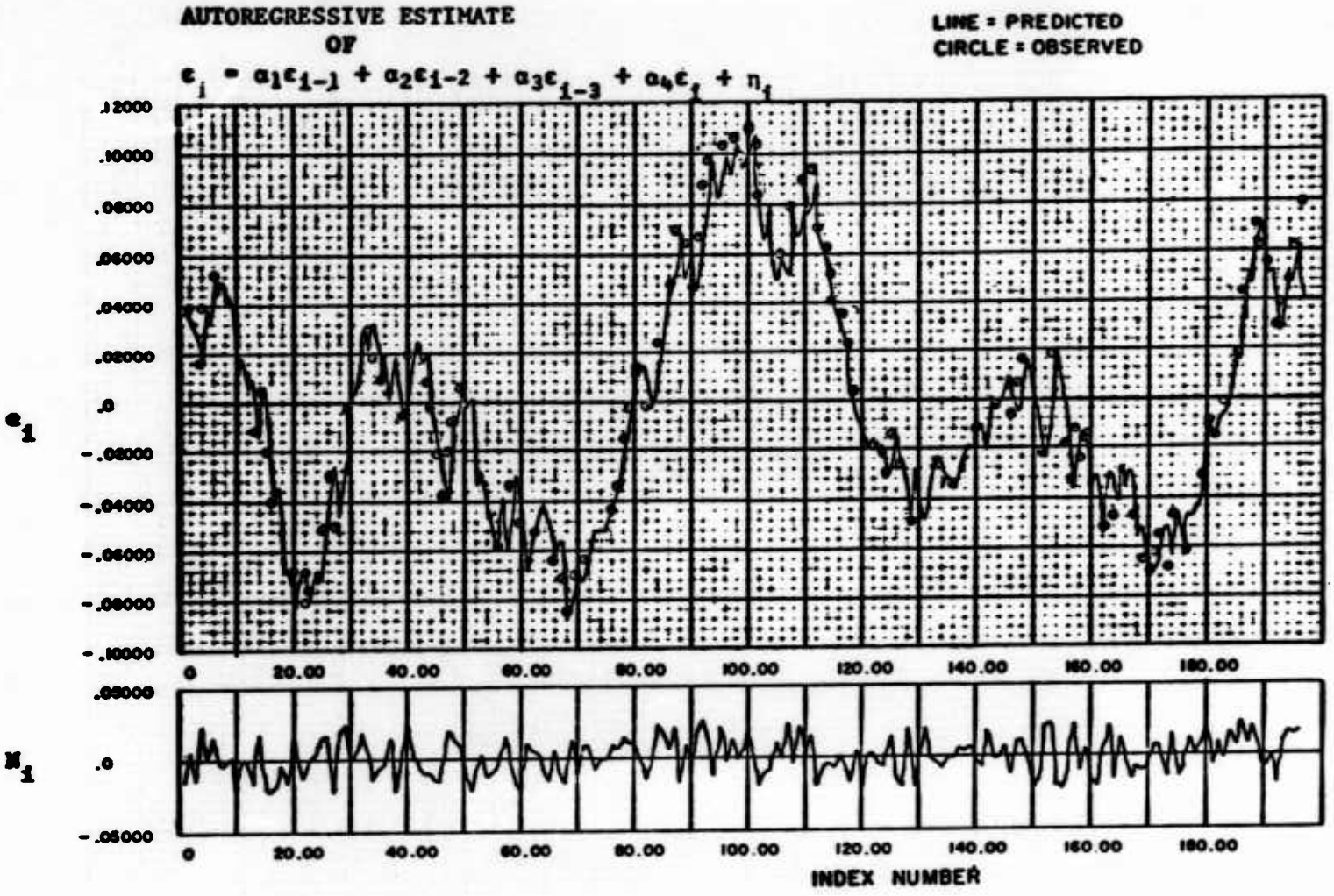


FIGURE 22 Estimate of autocorrelation function for first order MARKOV process with $\rho = \alpha_1 = 0.99$. The estimate at LAG k=1 is 0.97. Note the error in the estimate for large lags. RHO is not supposed to become negative for this particular process.



COEFFICIENT ESTIMATES:

A(0) =	.00005
A(1) =	.96848
A(2) =	.15390
A(3) =	.15523-
A(4) =	.01164-

FIGURE 23 Estimate of Markov process using 4th order autoregressive scheme. The regression coefficients indicate that the process is first order with $\rho = .96848$.

ANALYSIS OF VARIANCE FOR THE AUTOREGRESSION SCHEME

SOURCE OF VARIATION	D.F.	SUM OF SQUARES	MEAN SQUARES	F VALUE
DUE TO REGRESSION	4	0.42199	0.10550	570.1688
DEVIATION ABOUT REGRESSION . .	191	0.03534	0.00019	
TOTAL . .	195	0.45733		

VAR. NO.	MEAN	STD. DEVIATION	REG. COEFF.	STD. ERROR REG. COE.	COMP. T	PARTIAL CORR. COE.	S.S.	PROP. VAR. CUM.
1	-0.00172	0.04830	0.96848	0.07262	13.33615	0.69439	0.42096	0.92048
2	-0.00167	0.04838	0.15390	0.10022	1.53557	0.11043	0.00000	0.00001'
3	-0.00143	0.04875	-0.15523	0.09948	-1.56043	-0.11220	0.00102	0.00223
4	-0.00117	0.04909	-0.01164	0.07200	-0.16164	-0.01170	0.00000	0.00001
5	-0.00164	0.04843						
OCOMP. CHEDK ON FINAL COEFF.			-0.01164					

FIGURE 24 The F value indicates autoregressive scheme is a significant fit. The computed T value shows that the only significant regression coefficient is a_1. The amount of explained variation due to $a_1 e_i$ is 90% of the total variation.

The spectral density corresponding to ρ_k will be a band of frequencies about ω_o =W and the general form of the spectral density will be

$$f(\omega) = \frac{A(\omega^2 + b^2)}{W^2 + 2a\omega^2 + b^4}$$

where

$$A = \frac{k}{\pi}$$

$$b = \sqrt{k^2 + W^2} \tag{11.0.17}$$

$$a = k^2 - W^2$$

The continuous case of damped oscillatory covariance function is

$$R(\tau) = \sigma^2 e^{-\alpha|\tau|} \cos \omega_o \tau \tag{11.0.18}$$

Another example of random error would be the so-called moving average of a sequence of uncorrelated random variable. The stochastic model is

$$\varepsilon_j = \sum_{k=1}^{n} a^k \eta_{j-i}, \quad |a| < 1, \tag{11.0.19}$$

where the η's are independently distributed with zero mean. In this case it can be shown that in the limit as $n \to \infty$ the autocovariance function is of the form

$$R(k) = \frac{a^k}{1 - a^2} \tag{11.0.20}$$

which corresponds to a spectral density similar to (11.0.15) if $a = \rho$ and $\sigma^2 = \frac{1}{1 - a^2}$

The most general type of correlated noise is one where the correlation is varying and its graph has varying amplitude and periods. In this case the stochastic error model is

$$\varepsilon_t = \sum_{i=1}^{n} (a_i \cos \omega_i t + b_i \sin \omega_i t) \tag{11.0.21}$$

where the a_i and b_i are independent random variables with mean zero and variance $\sigma^2_{a_i} + \sigma^2_{b_i} = \sigma^2_i$. In this case the autocovariance function is

$$R(k) = \sum_{i=1}^{n} \sigma_i^2 \cos \omega_i k \tag{11.0.22}$$

and the spectral density function is a sequence of frequency bands made up of "impulses" corresponding to $\sigma_i^2/\Delta\omega$ at ω_i. The analytic form of $f(\omega)$ is

$$f(\omega) = \pi \sum_{i=1}^{n} \sigma_i^2 [\delta (\omega - \omega_i) + \delta(\omega + \omega_i)] \tag{11.0.23}$$

where δ is the Dirac delta function.

(11.0.21) is often the type of spectral density encountered in error analysis. An example of (11.0.23) is given in Figure 25.

Analysis of error in the frequency domain can be made more efficient from a computational standpoint if the fast Fourier transform is used (Reference 27). The autocovariance function can be obtained with less computation time by taking the fast Fourier transform, computing the modulus, smoothing, then taking the inverse fast Fourier transform.

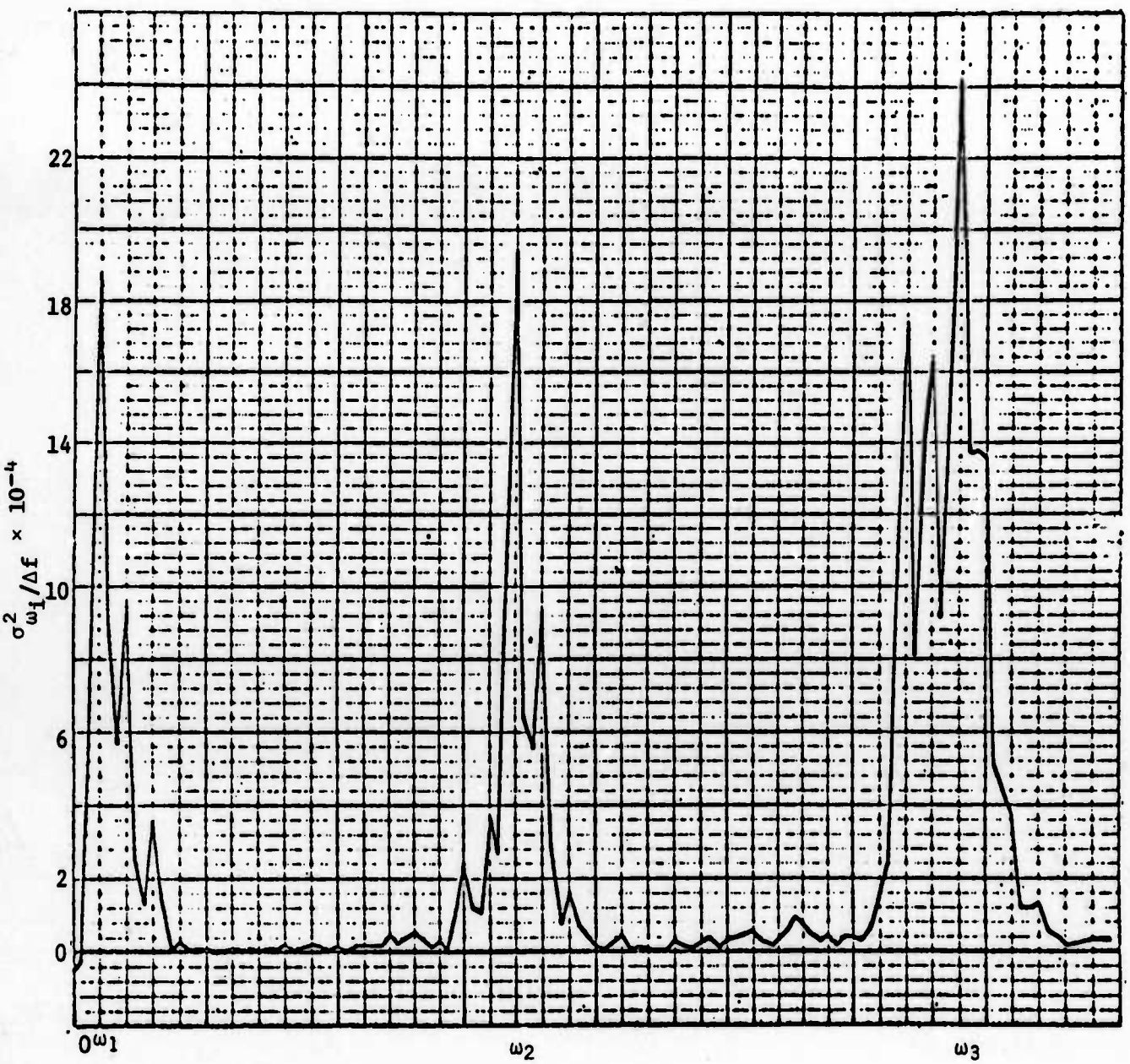


FIGURE 25 Estimate of the spectral density function for a stochastric process with random amplitudes and phases. The ordinate estimates the variance per frequency increment $\Delta f = \frac{1}{2M\Delta t}$.

Once the frequency content and autocovariance of the error ϵ_t have been estimated, one can filter the data to remove both high frequency random error and oscillatory systematic errors of moderately low frequency (say 0.05 hertz). In addition, the autocovariance function can be used to remove biased estimates of the variance of the error by extended least squares approximation. The analysis of data in the frequency domain can be utilized in conjunction with multiple regression and analysis of variance. For example, suppose the error is a stationary process of the Form (11.0.21) However, the significant frequencies ω_i are not necessarily harmonics and are not known. A spectral analysis would show significant frequency components and the corresponding squared amplitudes (see Figure 25). With k significant ω_i's selected from the spectrum, (11.0.21) could be approximated using multiple regression and solving for the a_i's and b_i's. An analysis of variance would then enable one to determine how good an approximation would be made on the expected values of a_i, b_i and their standard deviations σ_{a_i}, σ_{b_i}.

As an example a regression analysis was done in conjunction with error data corresponding to the spectrum estimates in Figure 25. The results of this analysis are given in graphical form in Figure 26. An analysis of variance indicated the regression analysis explained 82% of the total variation in the data. Finally, an estimate of the spectral distribution function corresponding to the spectral density function is given in Figure 27. The spectral distribution gives the estimate of variance as a function of frequency bands. From the graph we see that the total variance estimate corresponds to the intercept: $\hat{F}(0) - \hat{F}(\frac{1}{2\Delta t}) = 0.954$. The variance estimate over the band $\omega_A < \omega_3 < \omega_B$ is $F(\omega_B) - F(\omega_A)$ or $\sigma^2_{\omega_A-\omega_B} = 0.0365$.

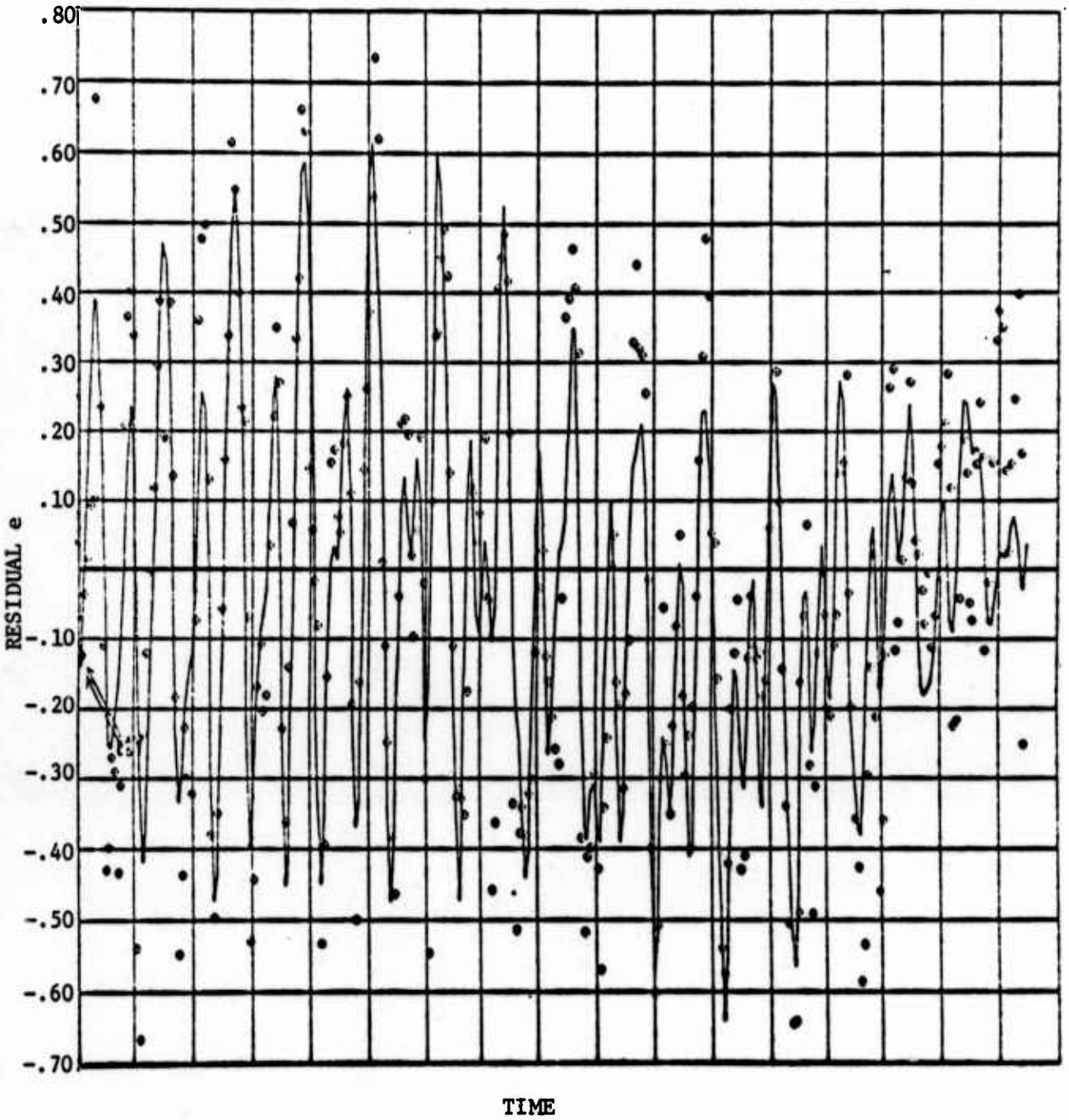


FIGURE 26 Regression analysis of data based on frequency information obtained from spectral density estimate (FIGURE 25). Line is predicted error while dots represent error measurements.

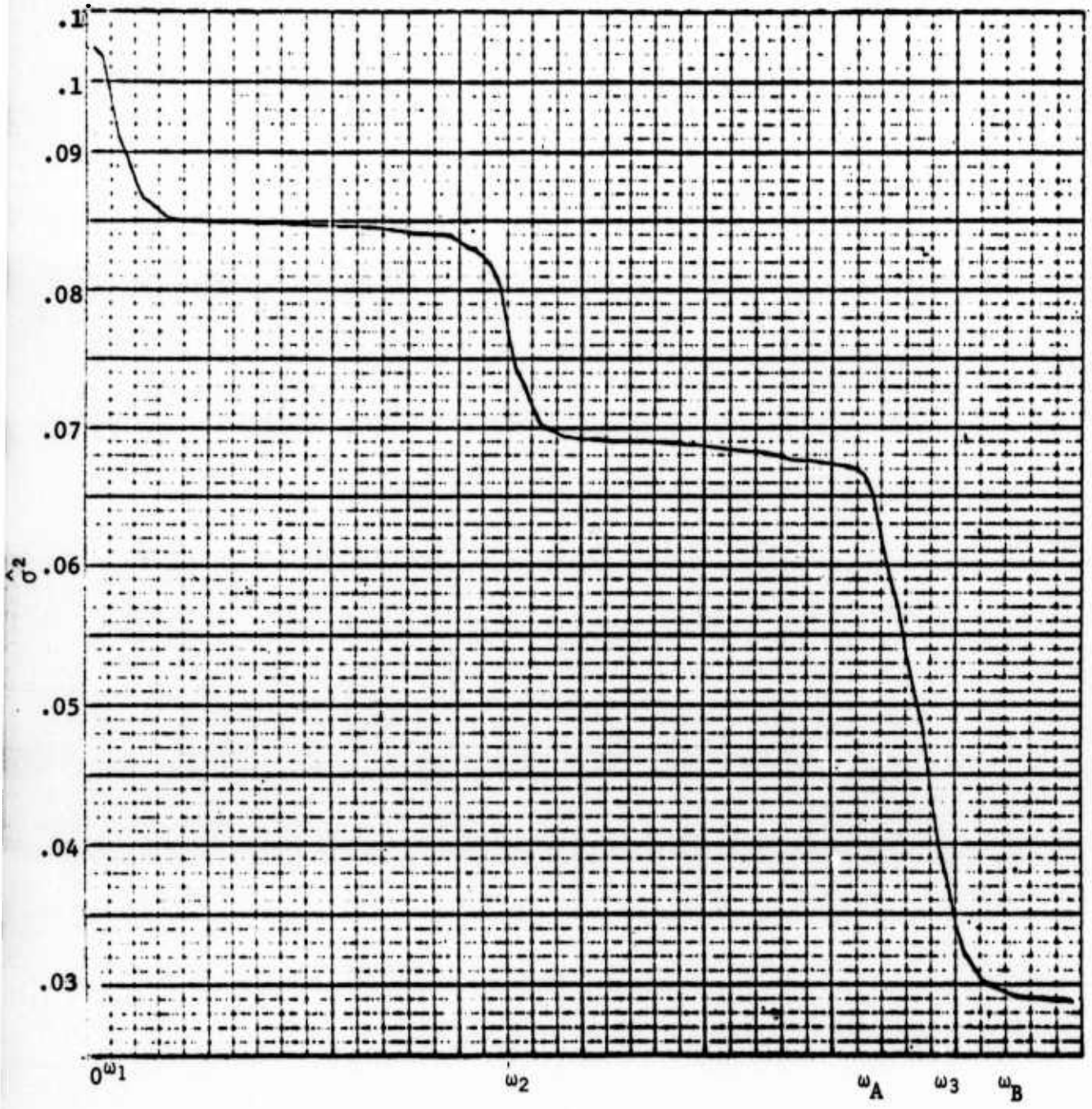


FIGURE 27 Estimate of spectral distribution function, F, corresponding to spectral density function in FIGURE 25.

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